PRINCETON UNIVERSITY

FPO | Dynamic Graphs

Sherrerd Hall 101; May 3, 2023 @ 4:30 PM

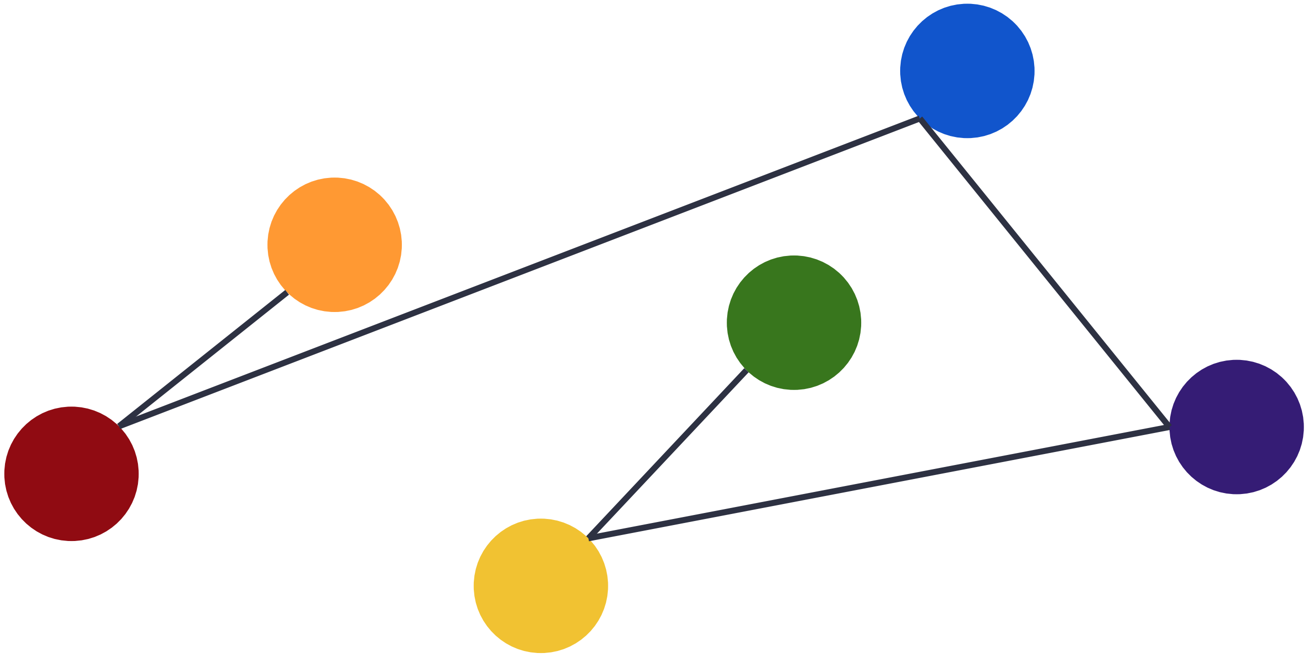
Devavrat Vivek Dabke

🙏 Acknowledgements □

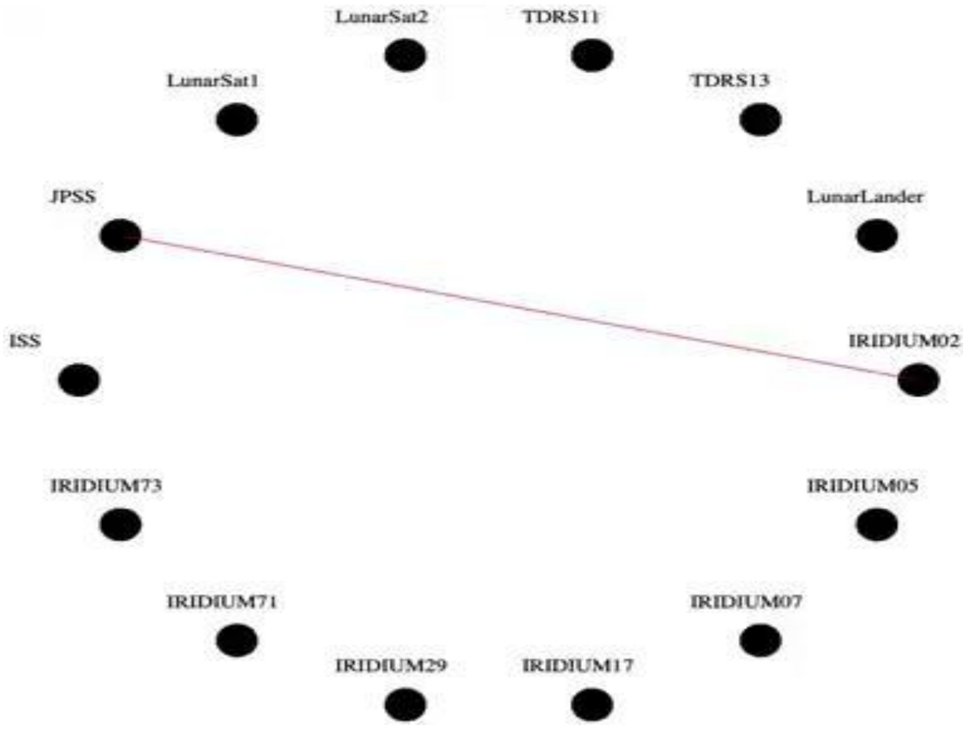
Overview

1. Brief mathematical introduction
2. Theoretical observations and conjectures
3. Application Domains
    a. Basketball (with ML)
    b. Space with **NASA**

🧮 Math Intro 🔢



A simple graph



A dynamic graph

Definition: Dynamic Graph

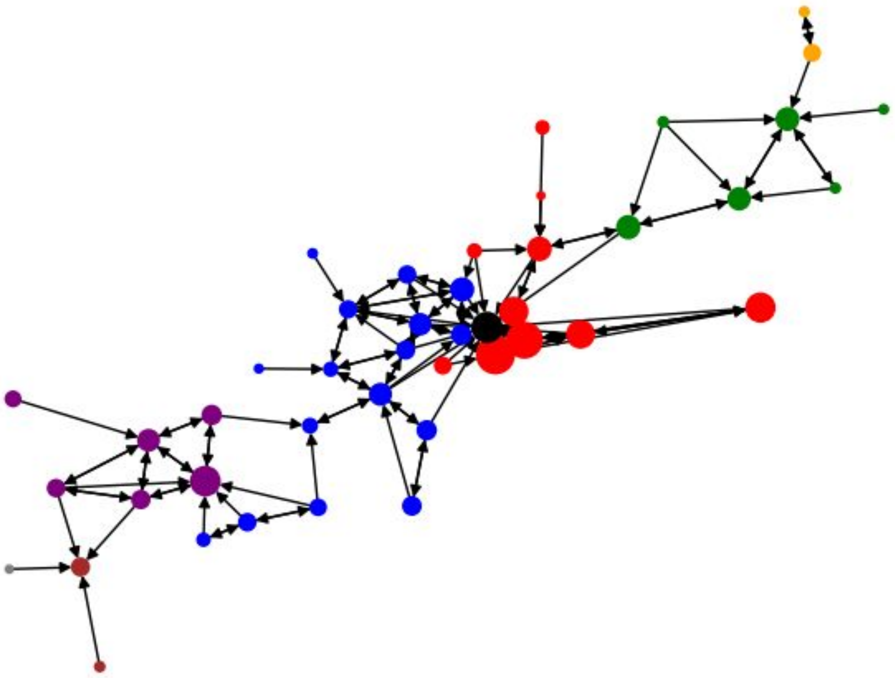
Dynamic graph $\mathcal{G} = (G_t)_{t \in \mathbb{T}}$ where

1. totally ordered indexing set $\mathbb{T}$
2. fixed finite vertex set V
3. edge set sequence $(E_t \subseteq V \times V)_{t \in \mathbb{T}}$
4. $G_t = (V, E_t)$

Dynamic Graphs

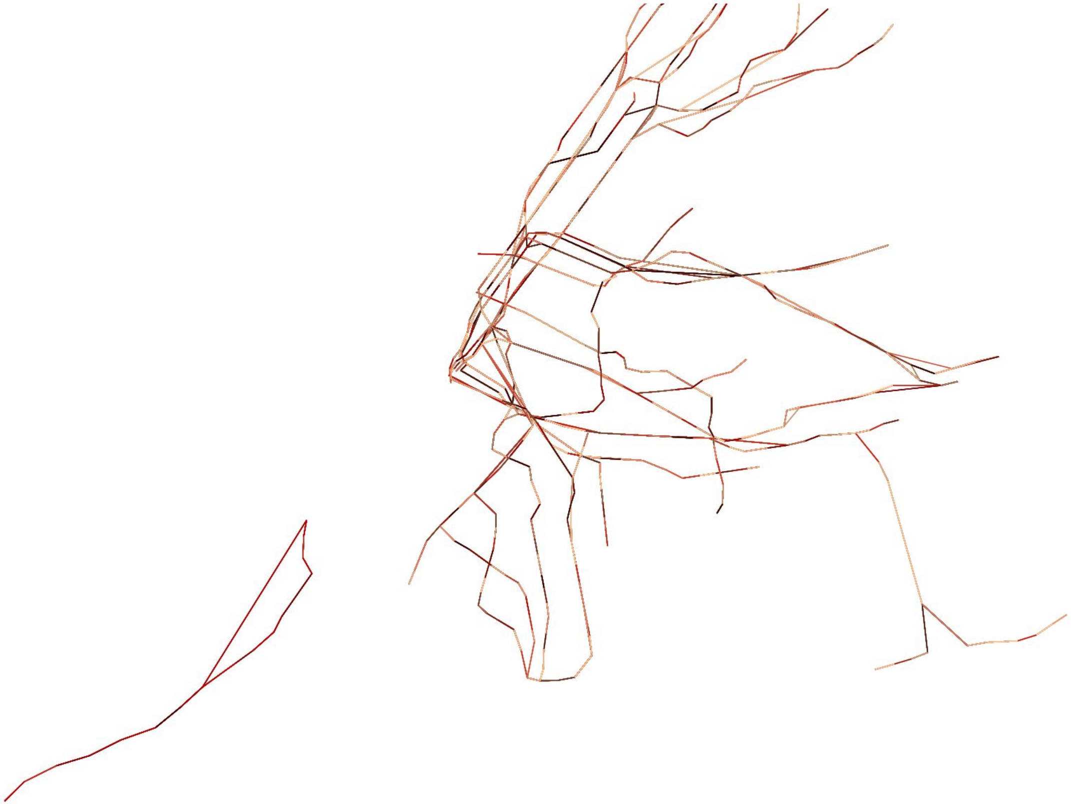
are **Everywhere**

- Commute networks
- Animal interaction networks
- Opinion dynamics
- Cell-cell signaling
- Social networks
- Satellite communication networks
- Basketball, sports
- Bird flocking



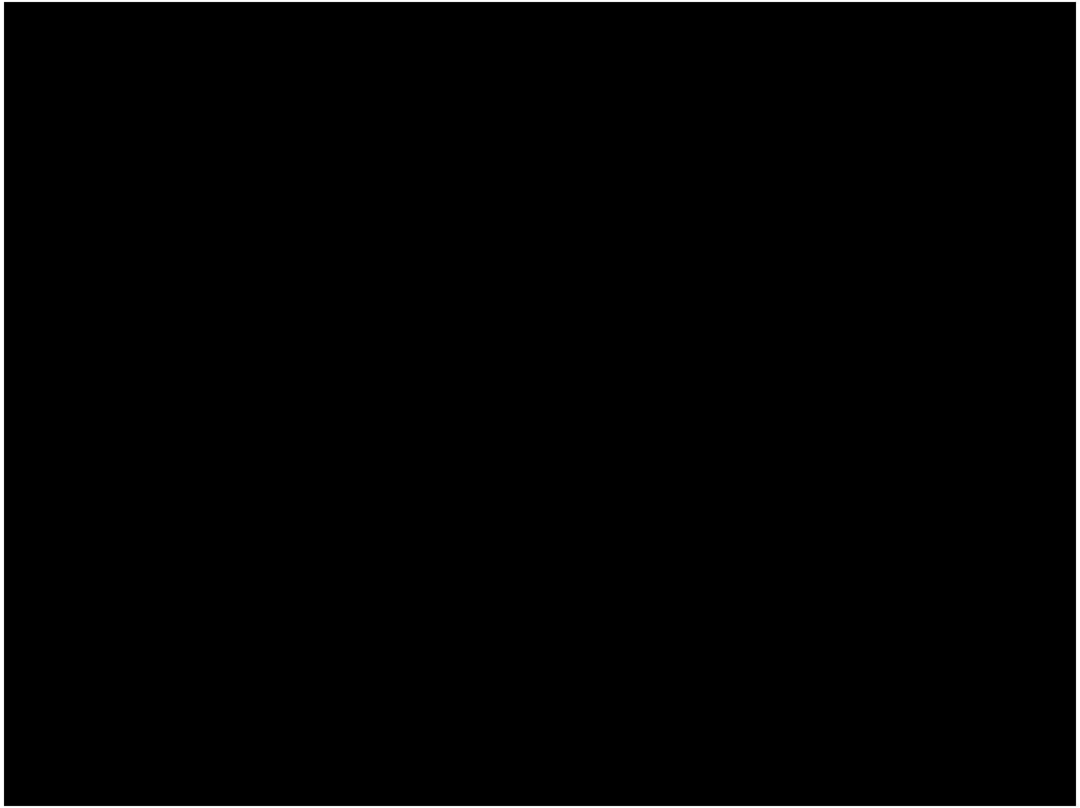
NYC Metro Area Commute Network

Dabke, Karntikoon, Aluru, Singh, Chazelle. *Network-augmented compartmental models to track asymp. disease spread*. (pre-print)



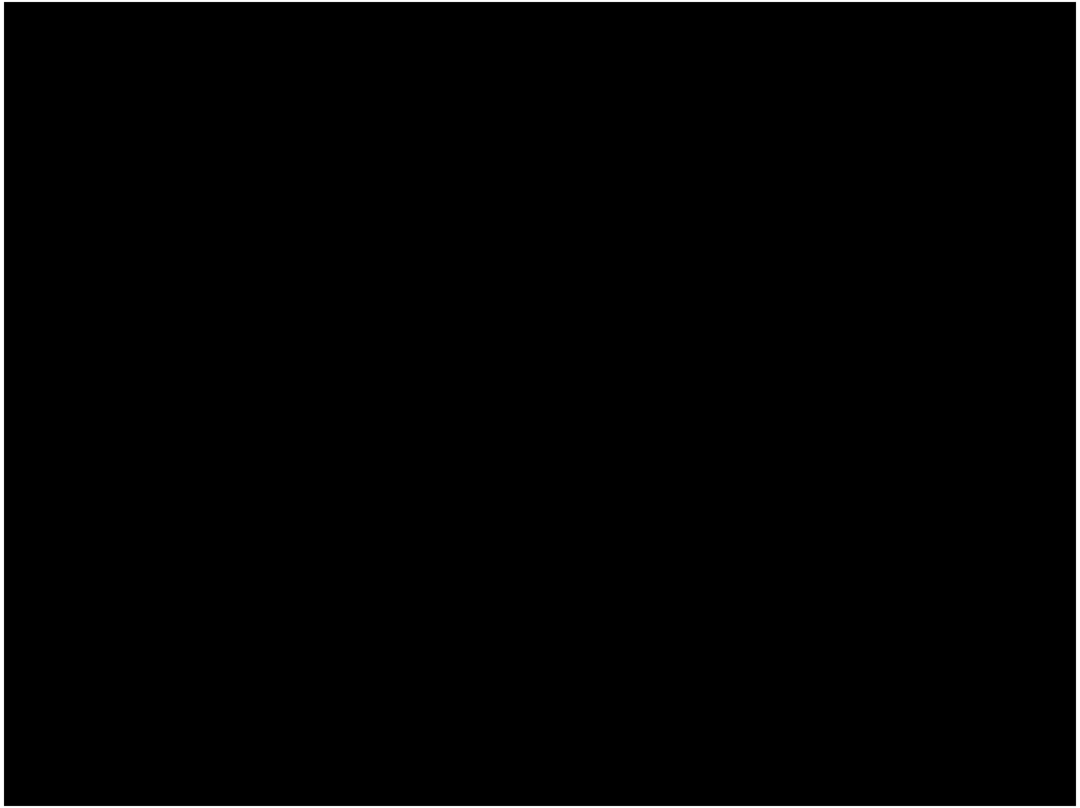
Transit Networks (GTFS)

Dabke, Green. *Analyzing transit networks with ideal routing machines*. (pre-print)

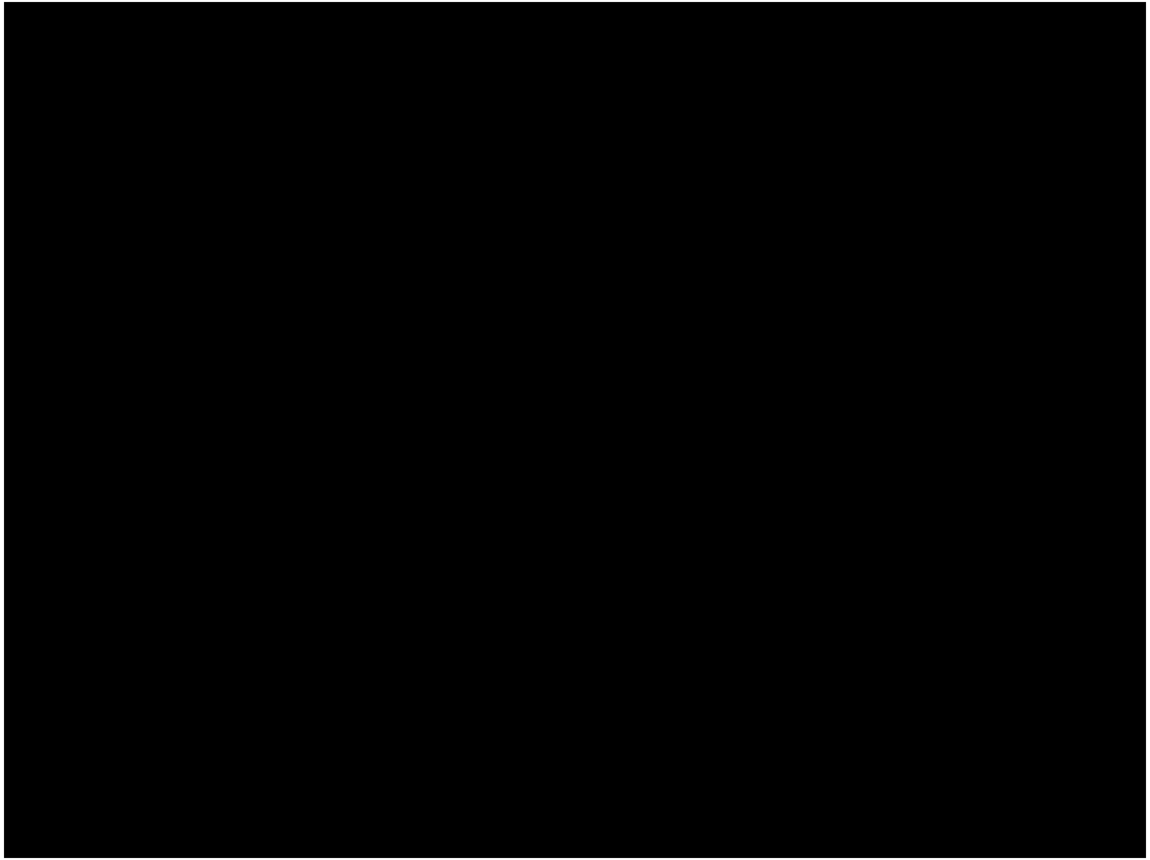


Animal herding networks

Dorabiala, Dabke, et al. *Spatiotemporal* k-*means*. (pre-print)



Embryos: spatial and chemical connectivity

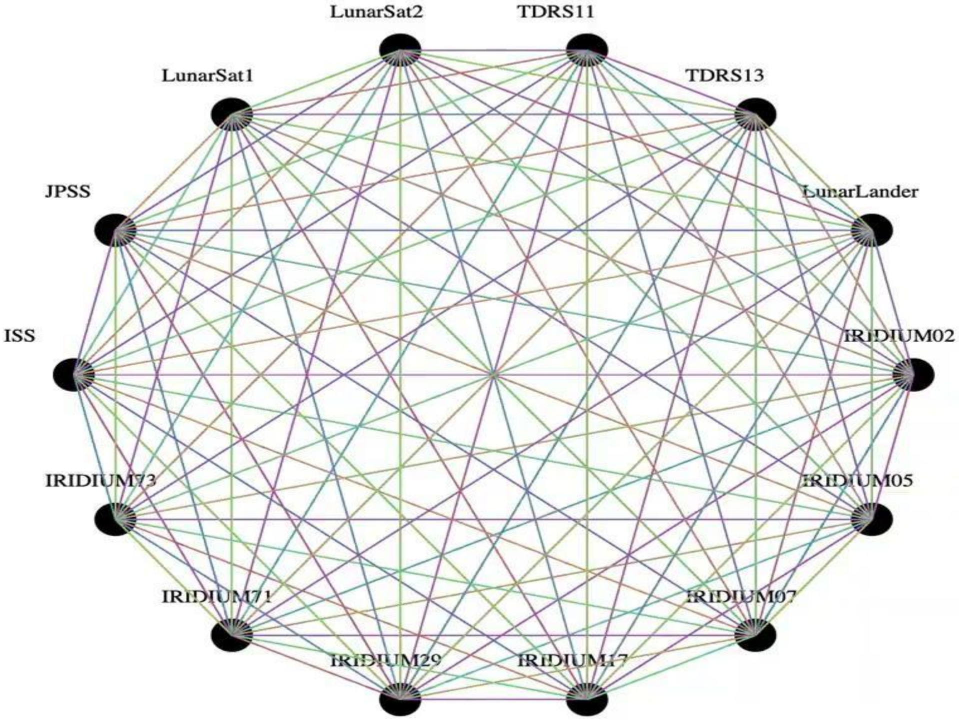


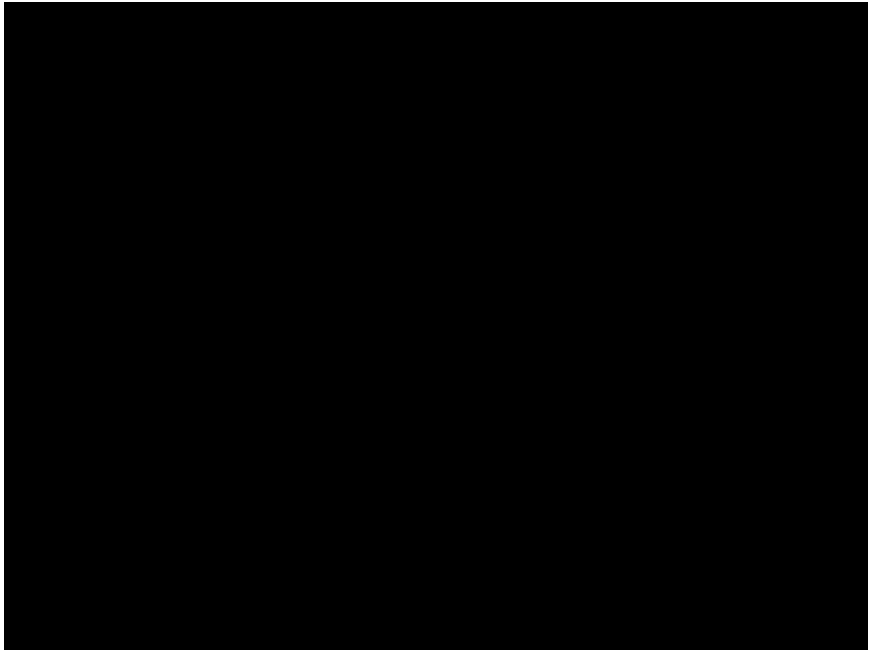
Basketball

Dabke, Chazelle. *Extracting semantic information from dynamic graphs of geometric data*.
Dabke, Taylor. *Play classification in basketball networks*. (internal publication; pre-print)

CLASSIFIED

NASA: Satellites





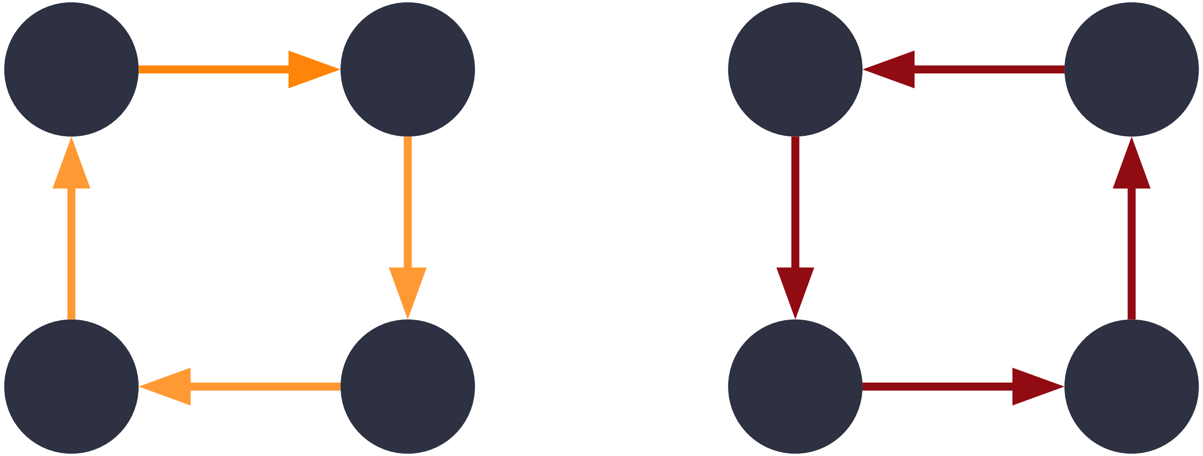
NASA: Satellites

Cleveland, Dabke, et al. *Introducing tropical geometric approaches to delay tolerant networking optimization.*
Hylton, Dabke, et al. *A survey of mathematical structures for lunar networks.*

Some Problems

Problem

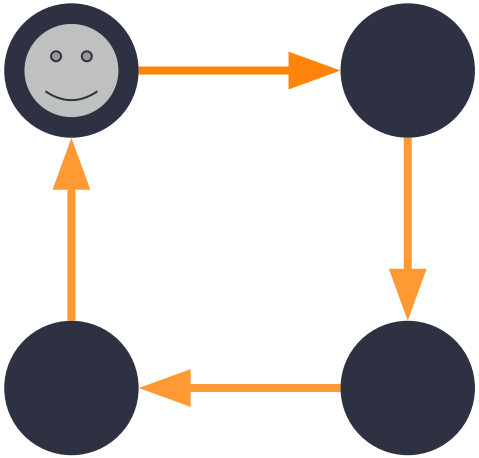
Local ≠ Global



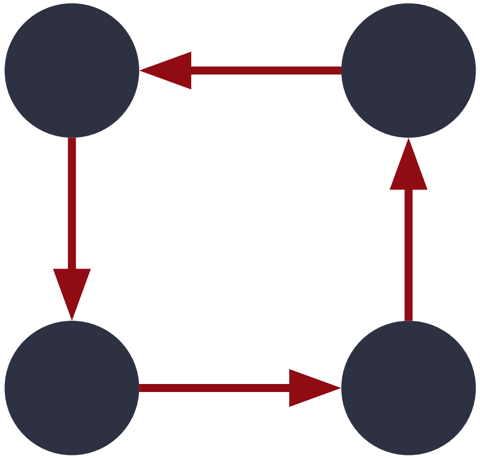
t = 1, 3, 5, ...

t = 2, 4, 6, ...

The alternating cycle: a discrete-time sequence

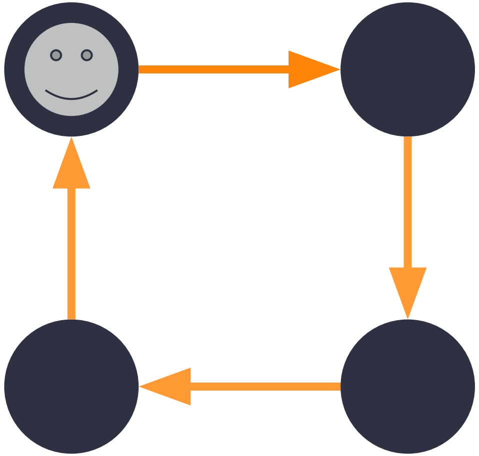


t = 1, 3, 5, ...

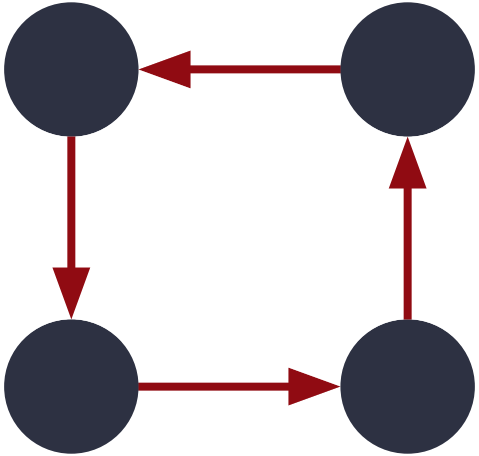


t = 2, 4, 6, ...

Dynamical system: move one edge at each time step

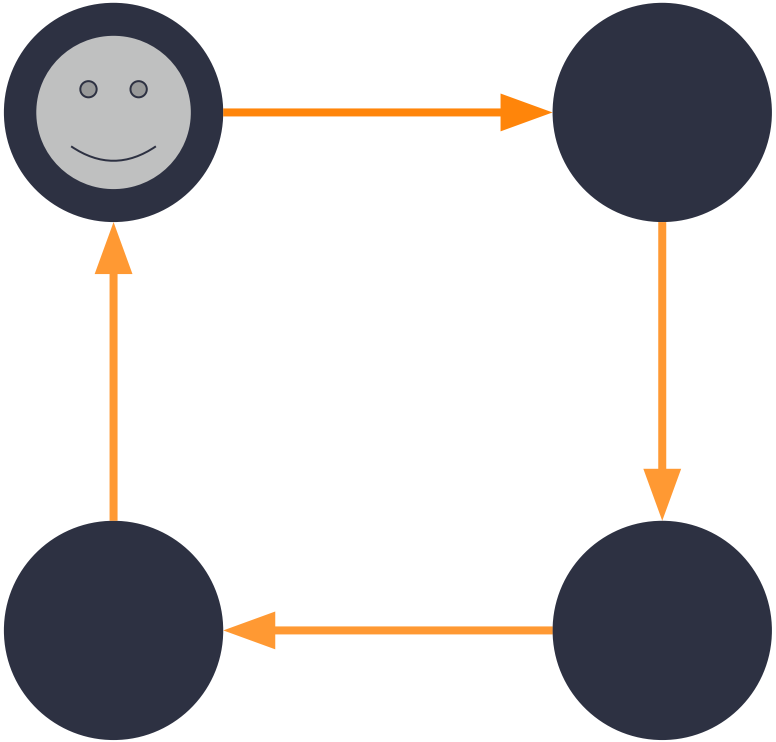


t = 1, 3, 5, …



t = 2, 4, 6, …

Fact: max diameter is number of vertices (if connected)



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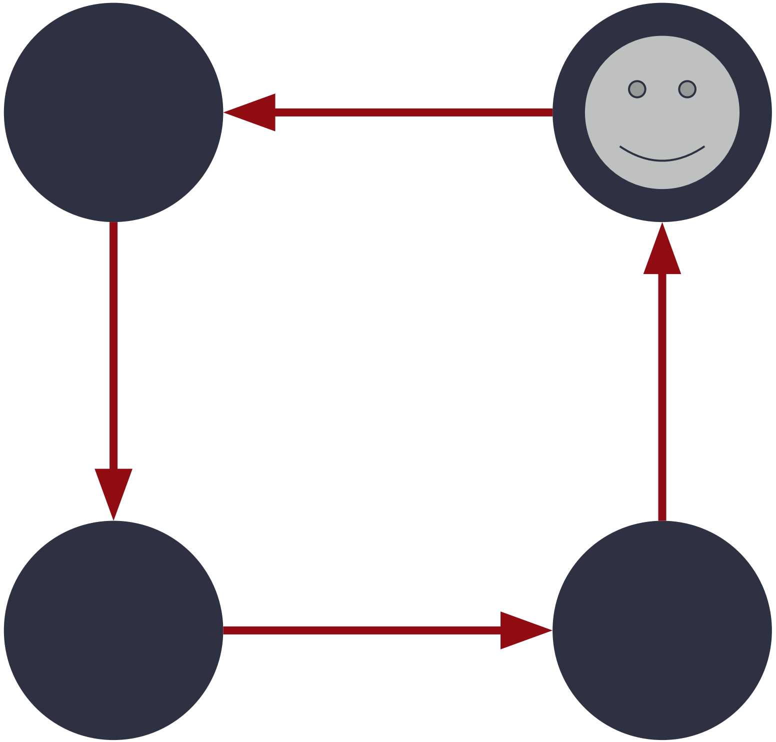
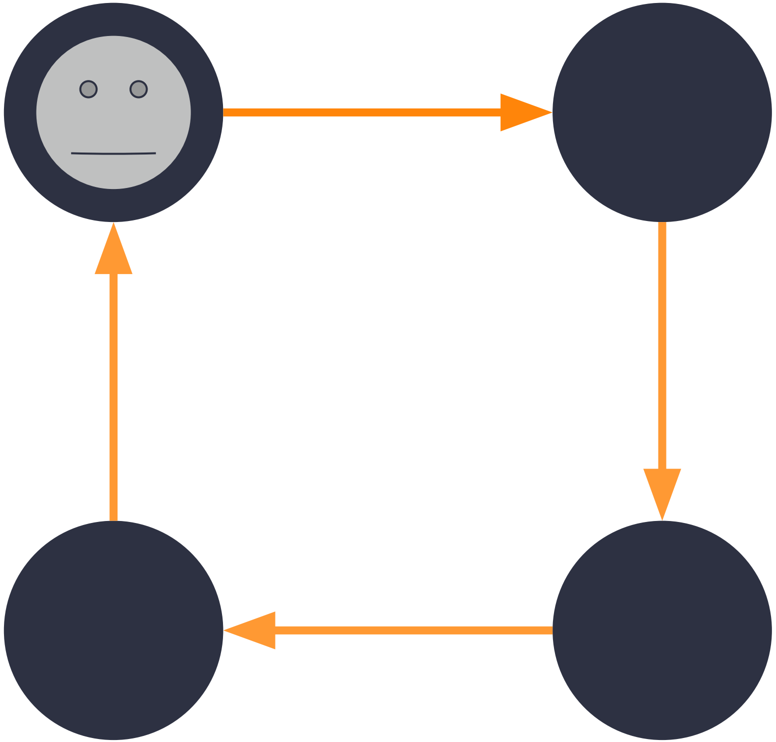


t = 2



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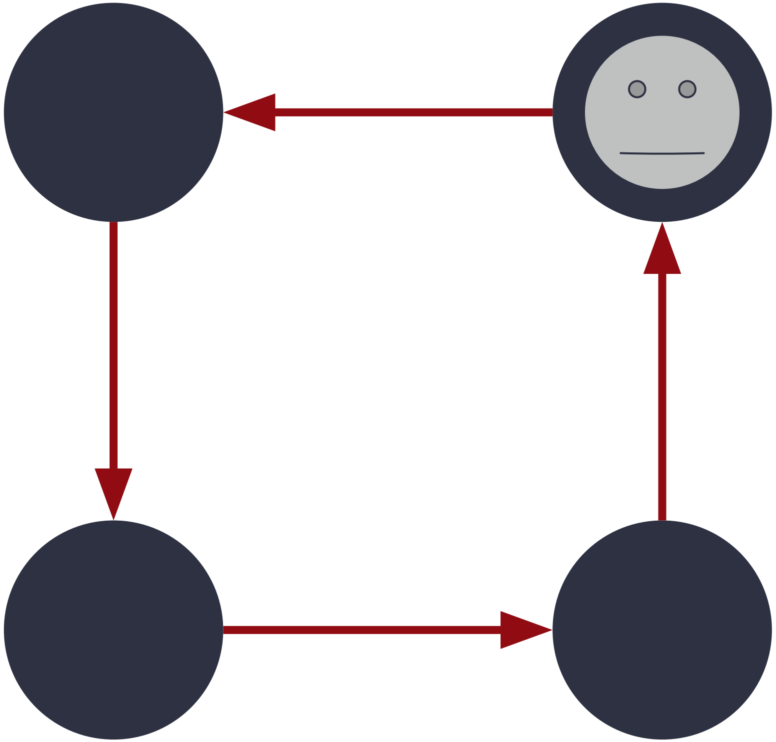
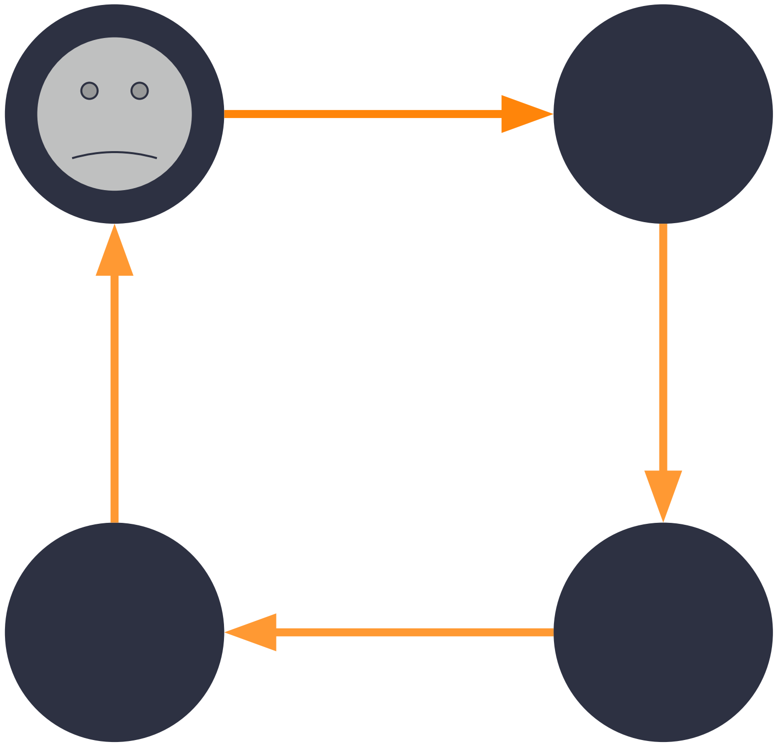
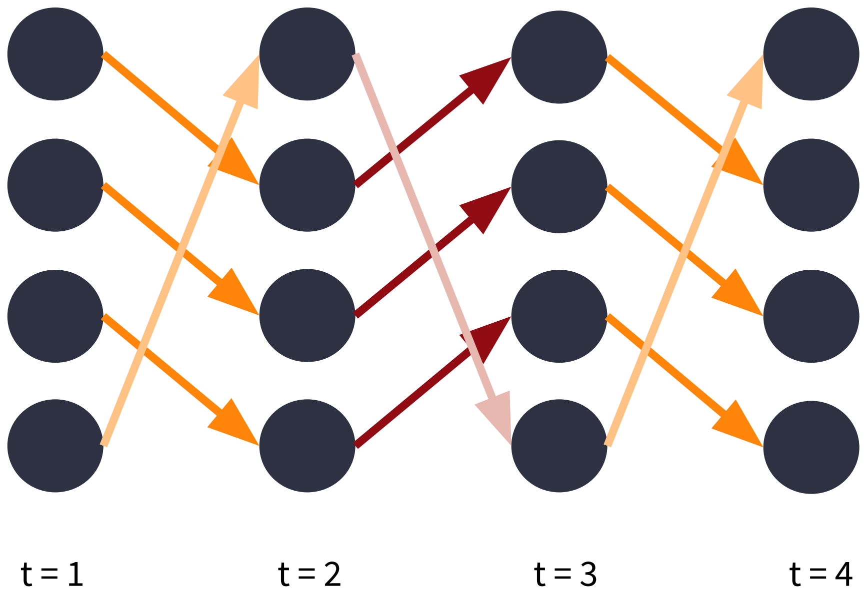


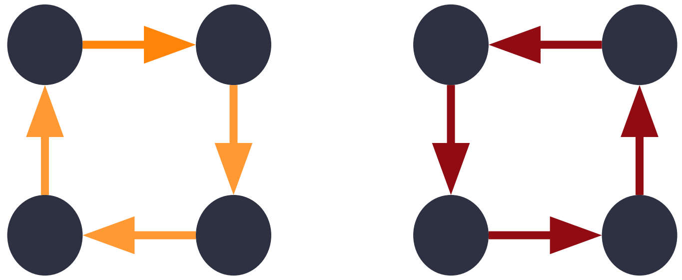
t = 4



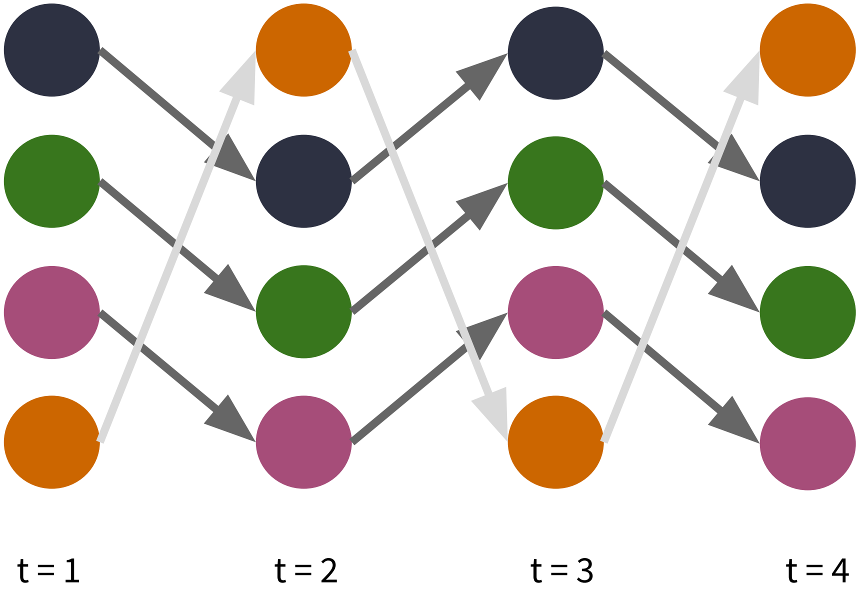
t = 5

Dynamically disconnected

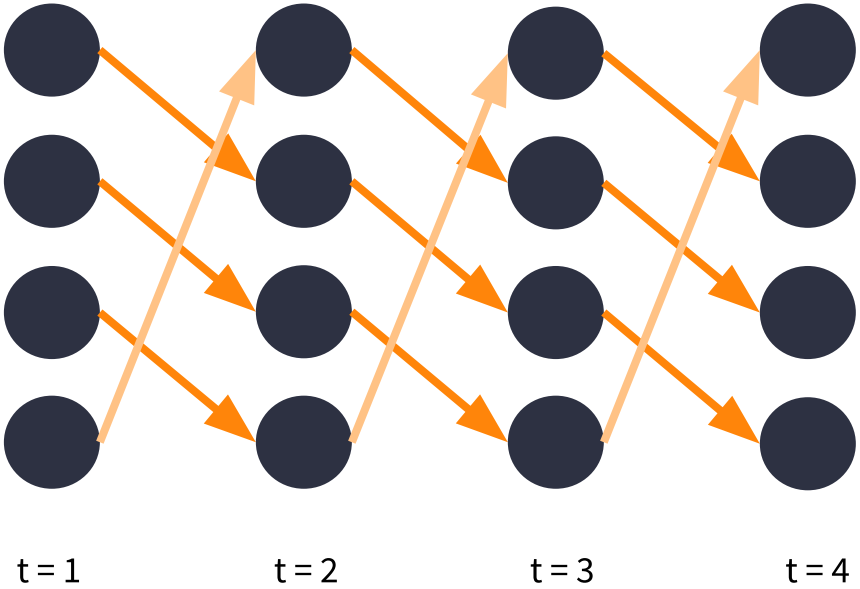




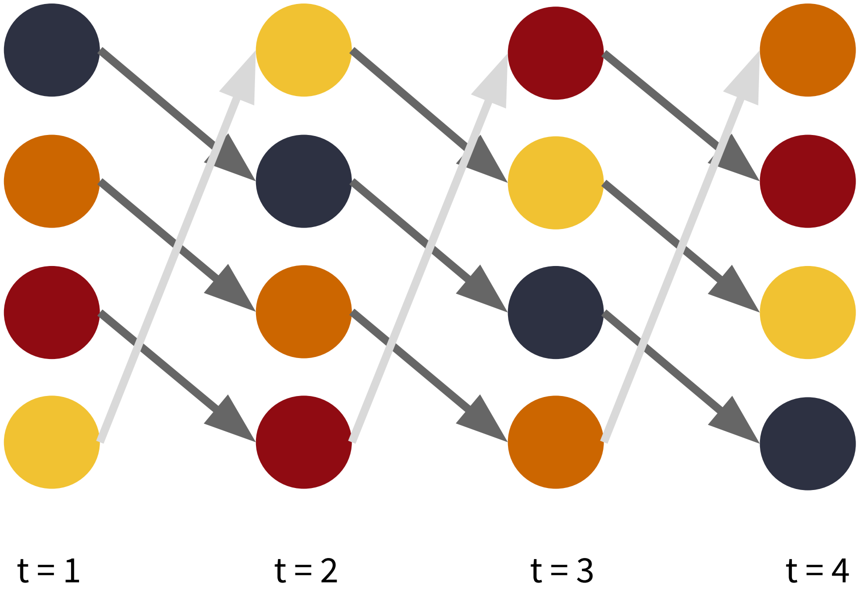
Idea: time-expanded graph



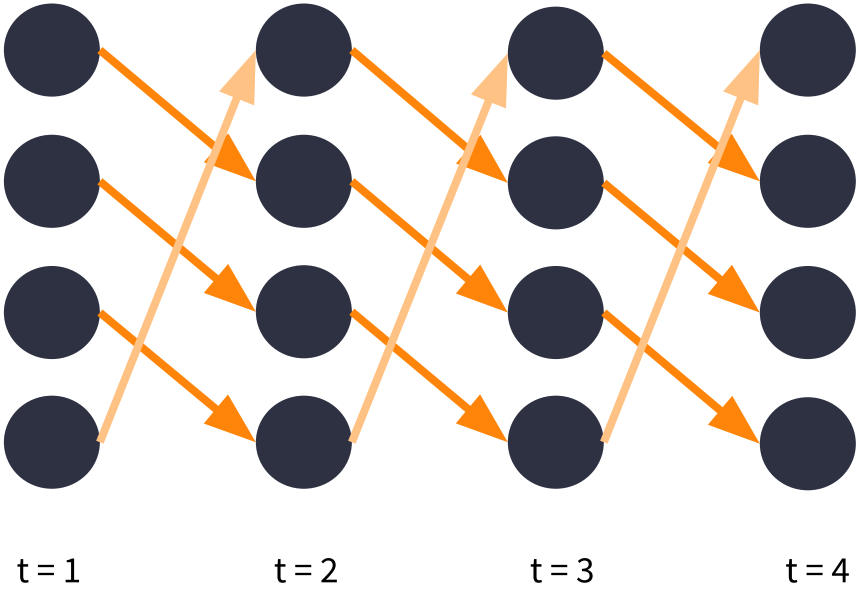
Observation: disconnected => dynamically disconnected?

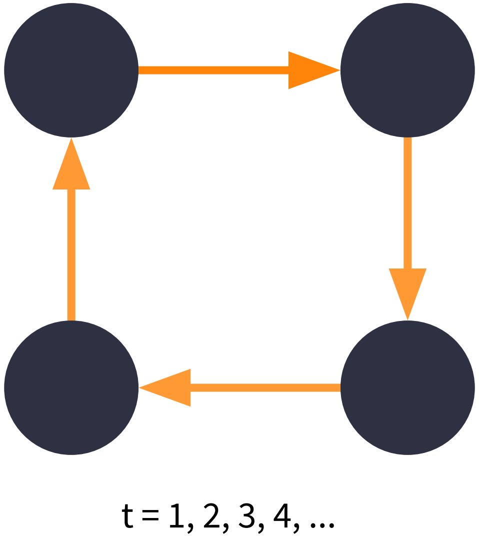


This one?

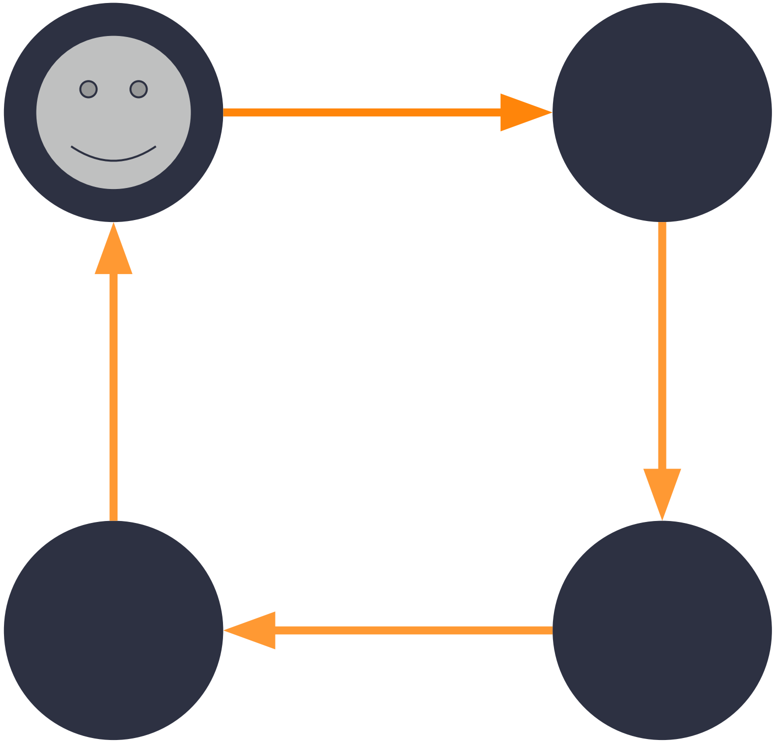


Disconnected again

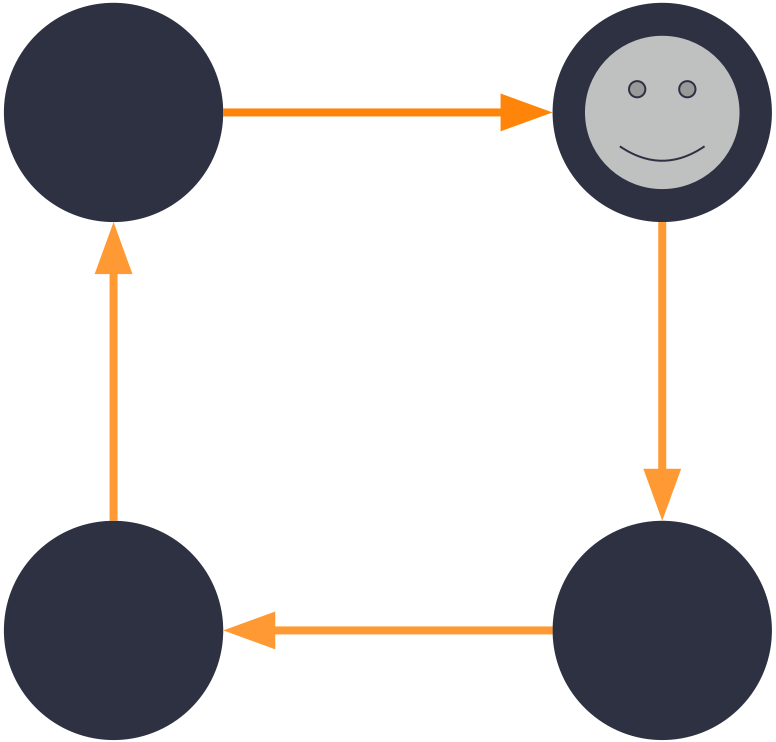




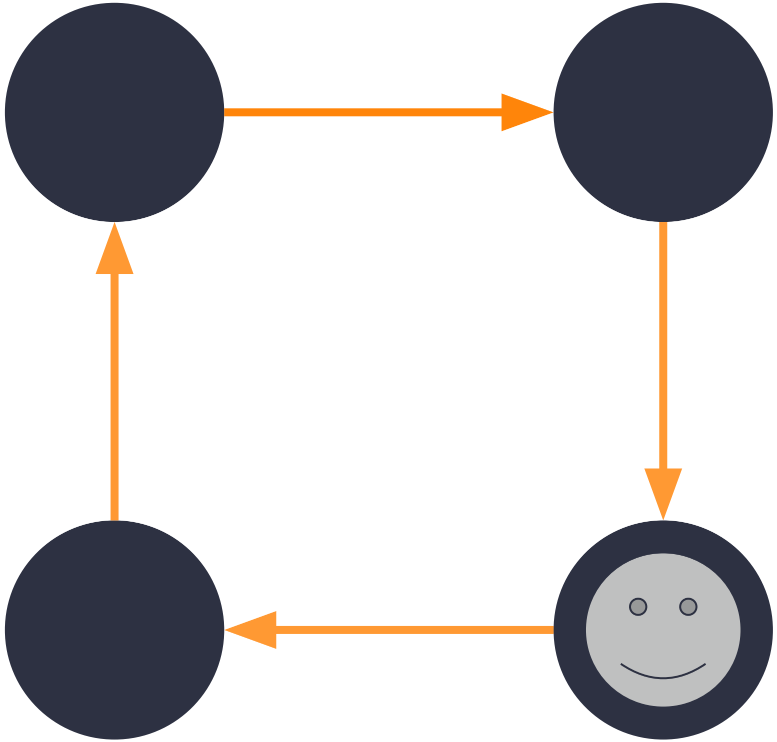
Time-expanded fixed cycle



t = 1



t = 2



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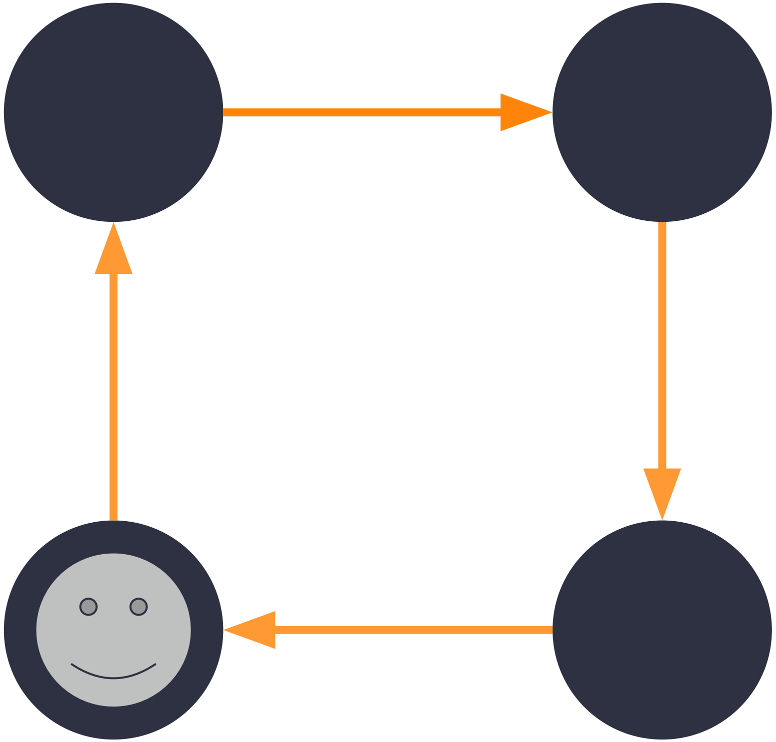
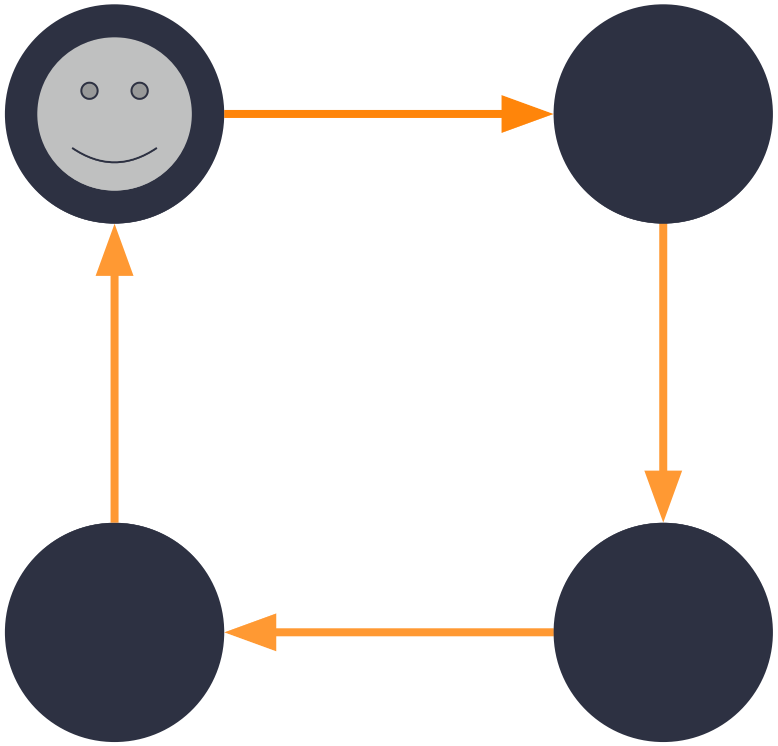


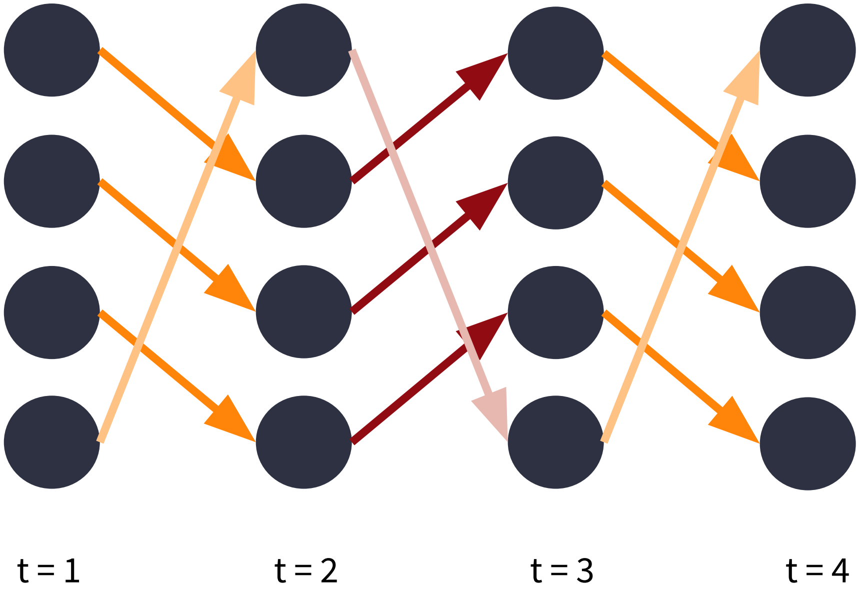
t = 4



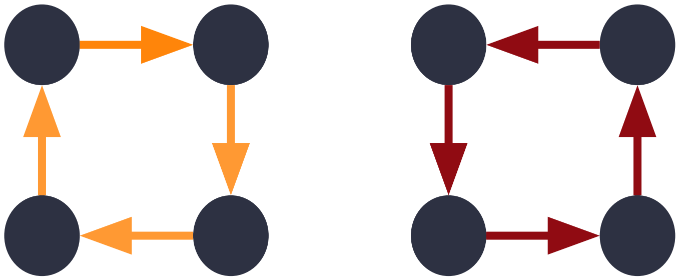
t = 5

Problem

$$\nu = n * t$$



too many nodes



Many Problems Don't "Just Work"

- Can we relate static and dynamic properties?
- How do we recover classical algorithms?
- Is there an efficient way to do all this?

🤓 Theory 💡

Two Orthogonal Dichotomies

Length

finite vs. infinite

Discretization

discrete-time vs. continuous-time

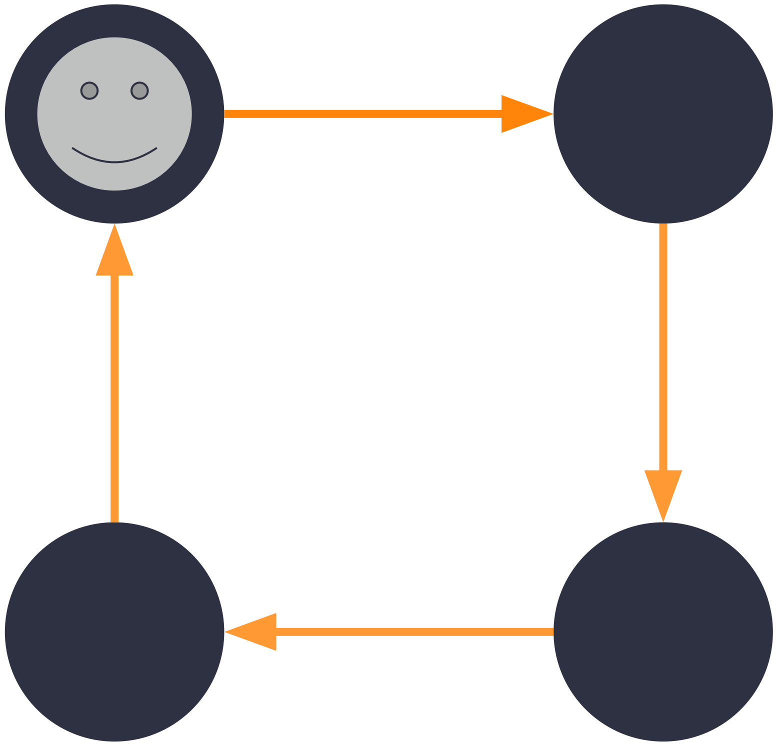
Three Results

I. Define dynamic analogs of static properties and relate them [2]
II. Graph summarization: make representations more efficient [1]
III. Connect to algebraic topology [1, 2]

1. Cleveland, Dabke, et al. *Introducing tropical geometric approaches to delay tolerant networking optimization.*
2. Hylton, Dabke, et al. *A survey of mathematical structures for lunar networks.*

Dynamic Connectivity

Journeys are Dynamic Paths



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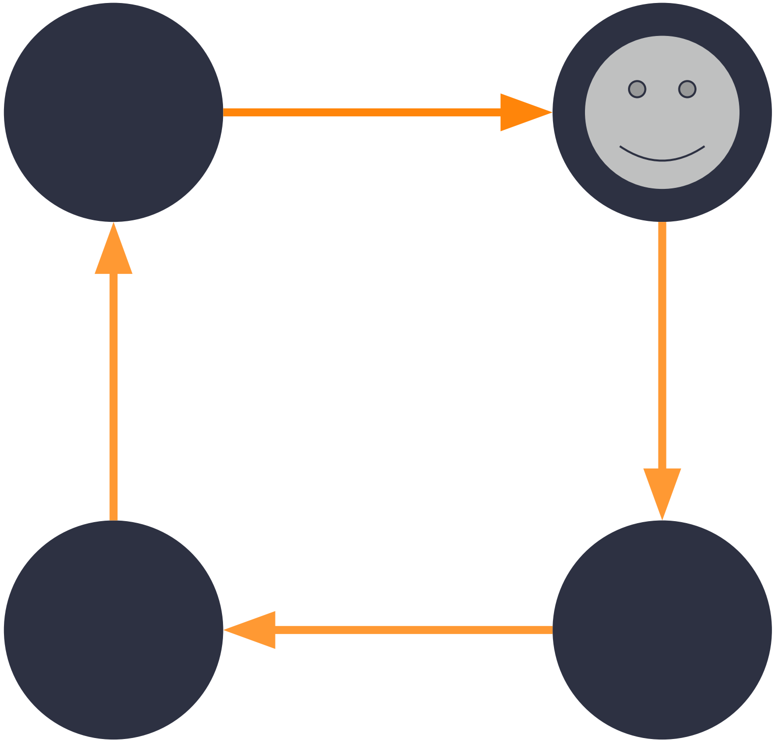
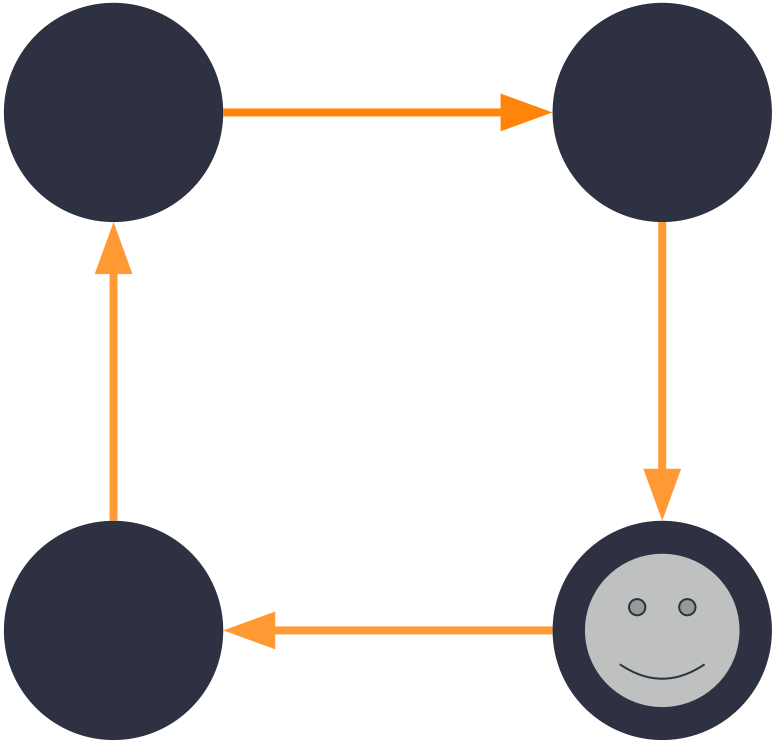


t = 2



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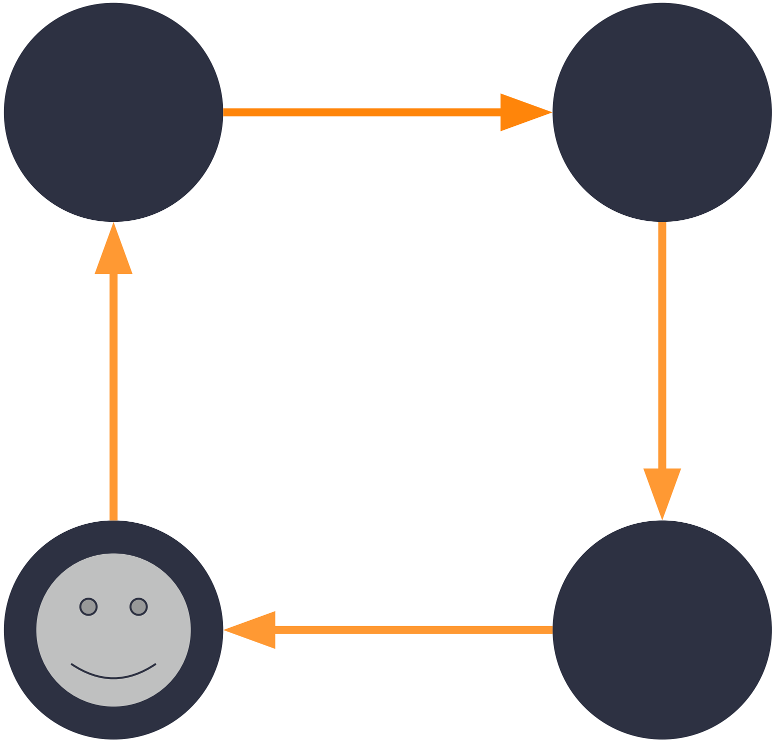

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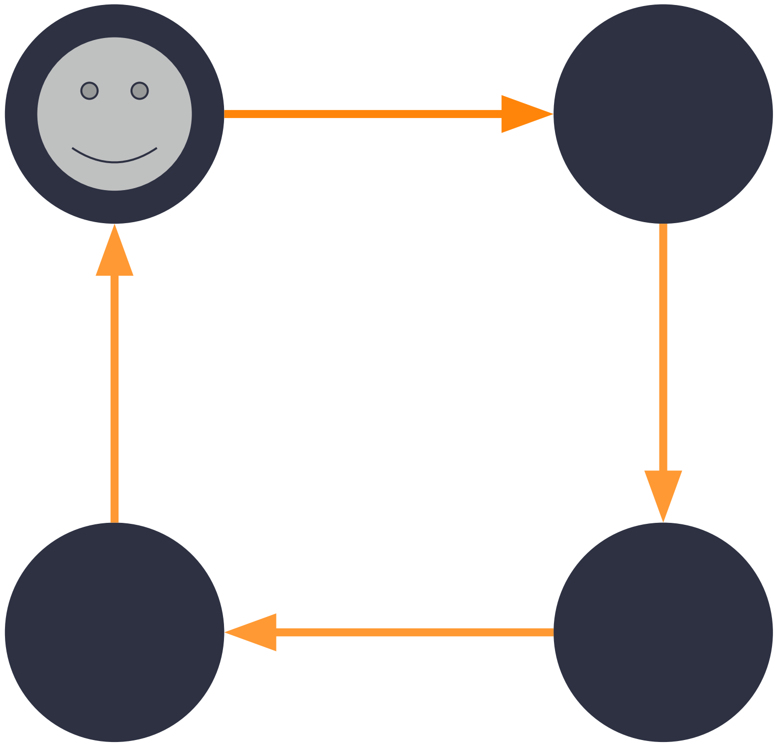


t = 5

Definition: Dynamic Diameter

- discrete-time (infinite or finite)
- defined at each timestep
  - it’s a sequence, not a number
- max of
  - shortest journey between all vertex pairs

Definition: Dynamically Connected

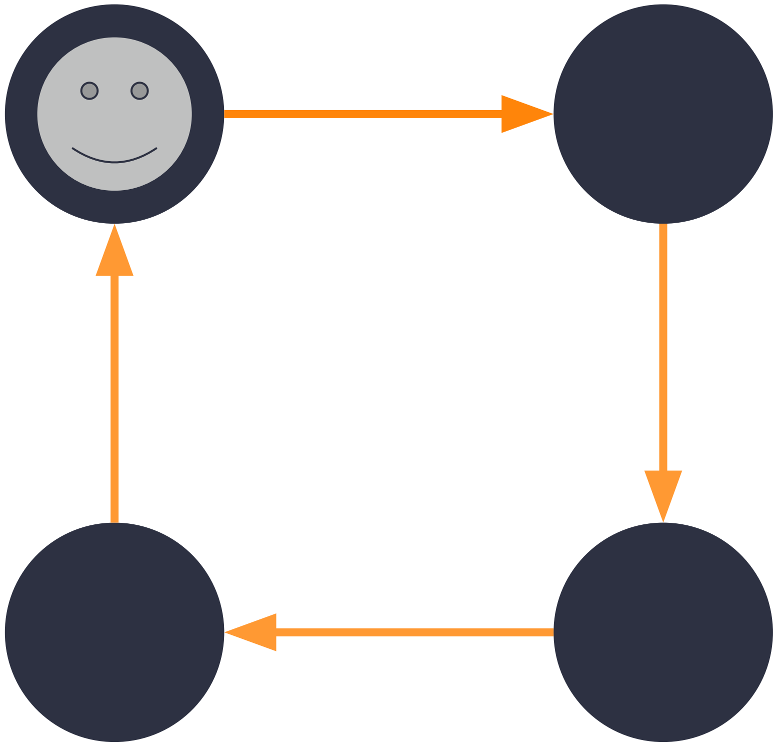
- *Connected* if diameter is always finite
- *Uniformly connected* if bounded

Proposition (We Didn’t Mess Up I)

If at any time a vertex has no outbound edges, the graph is dynamically disconnected.

Proposition (We Didn’t Mess Up II)

For a fixed dynamic sequence: diameter equals that of base graph.



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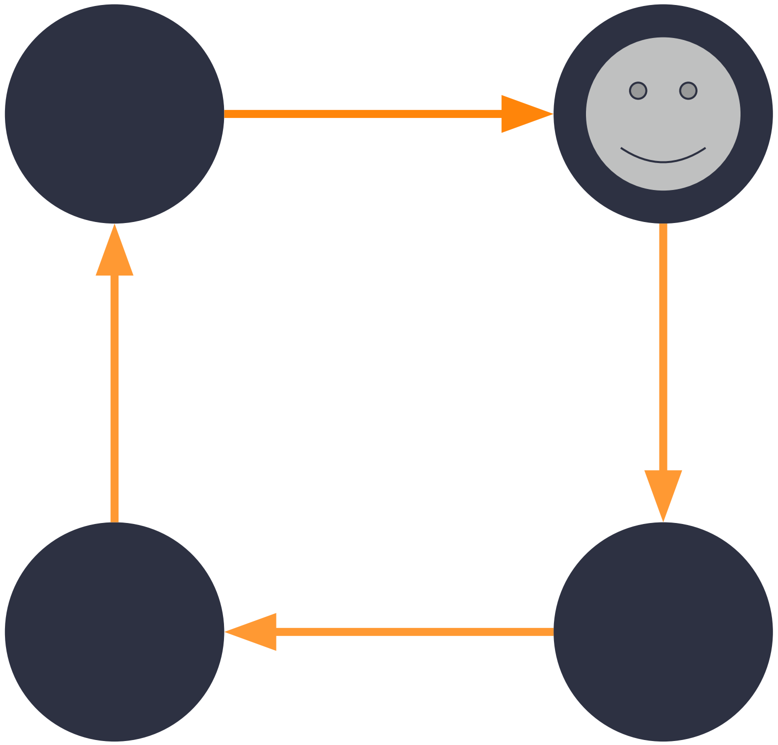

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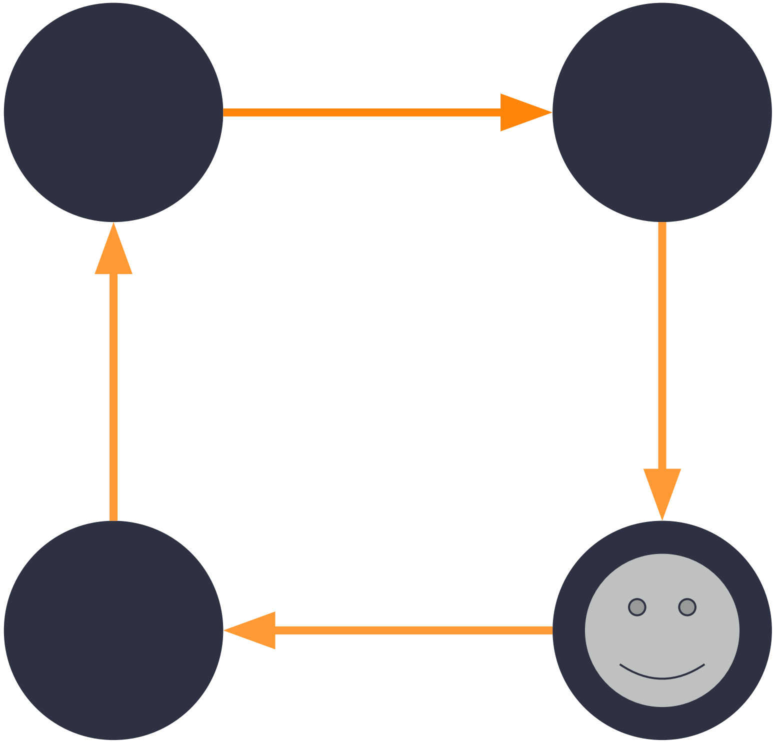

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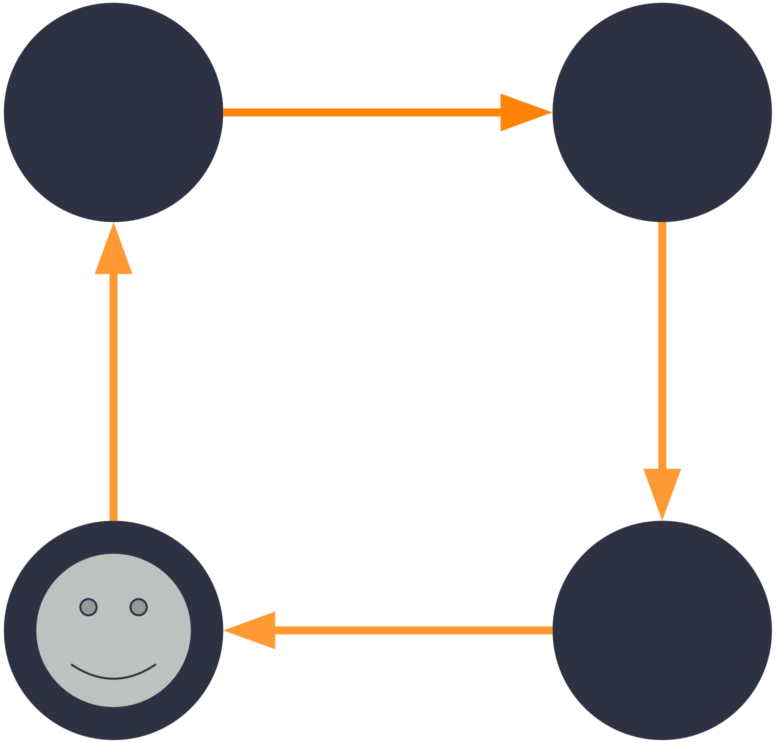

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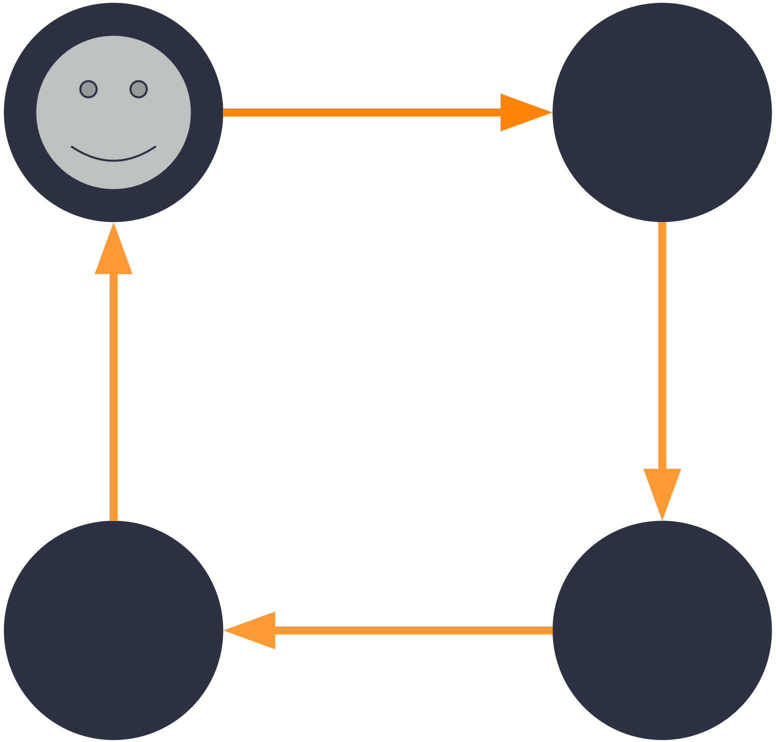


t = 5

Question

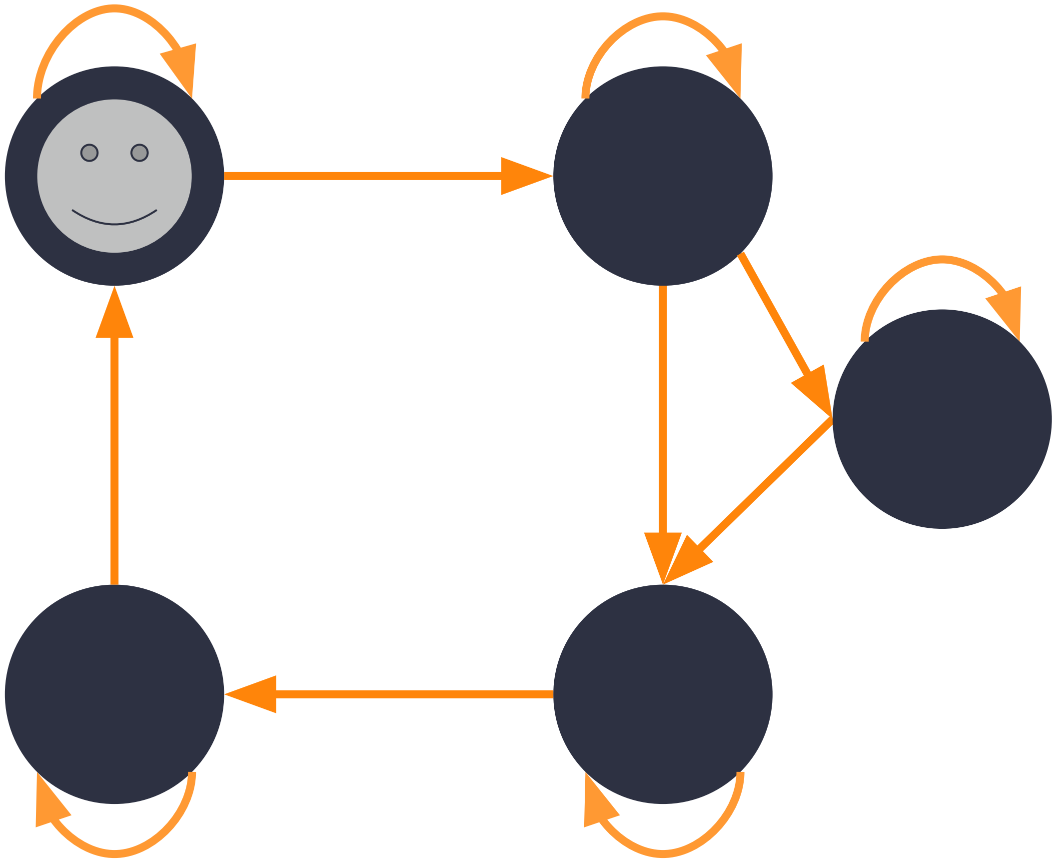
When does static imply dynamic?

Result: Self-Loops are Sufficient

Static connectivity implies dynamic connectivity if self-loops.

Other notes:

- Stronger (but more technical): weak monotonicity is sufficient.
- Uniform bound: number of vertices



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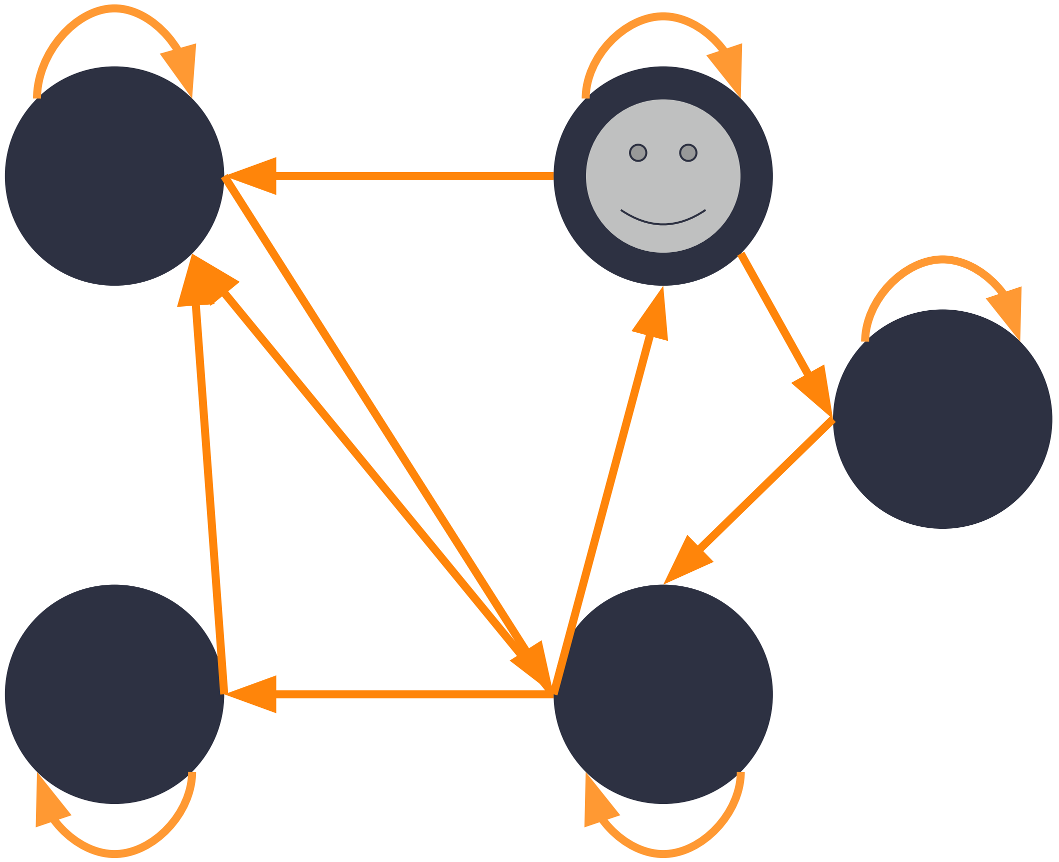
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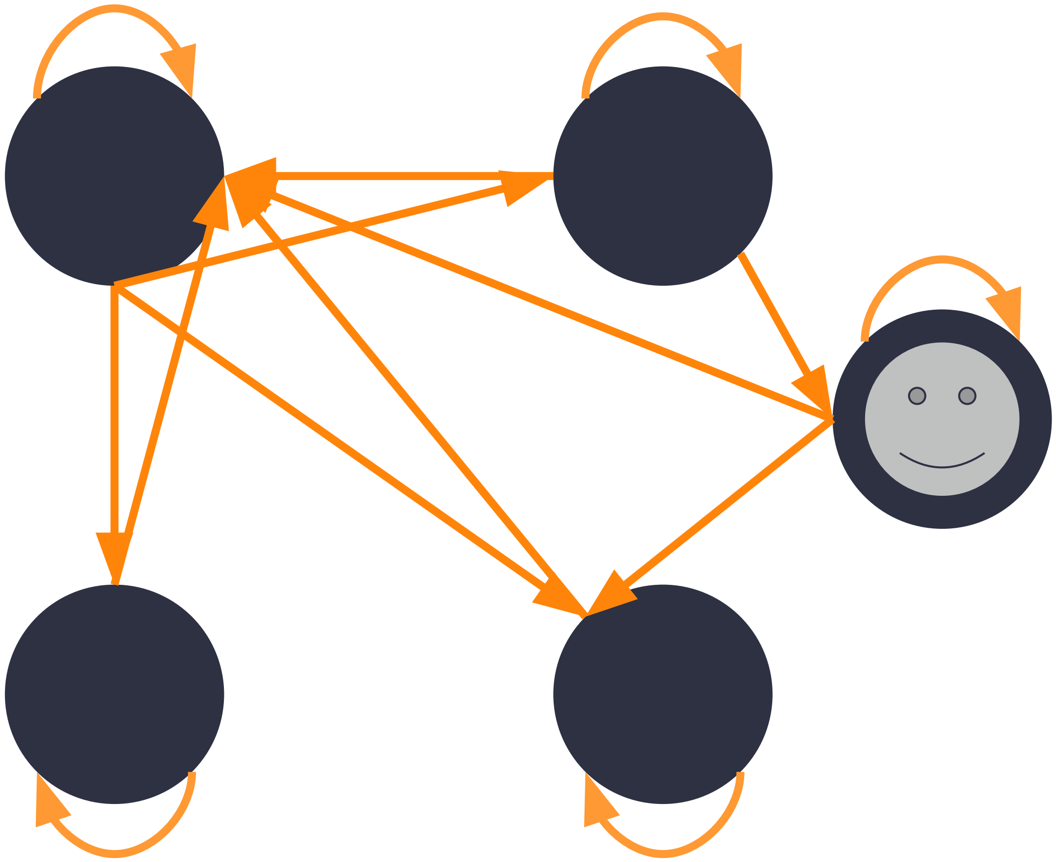

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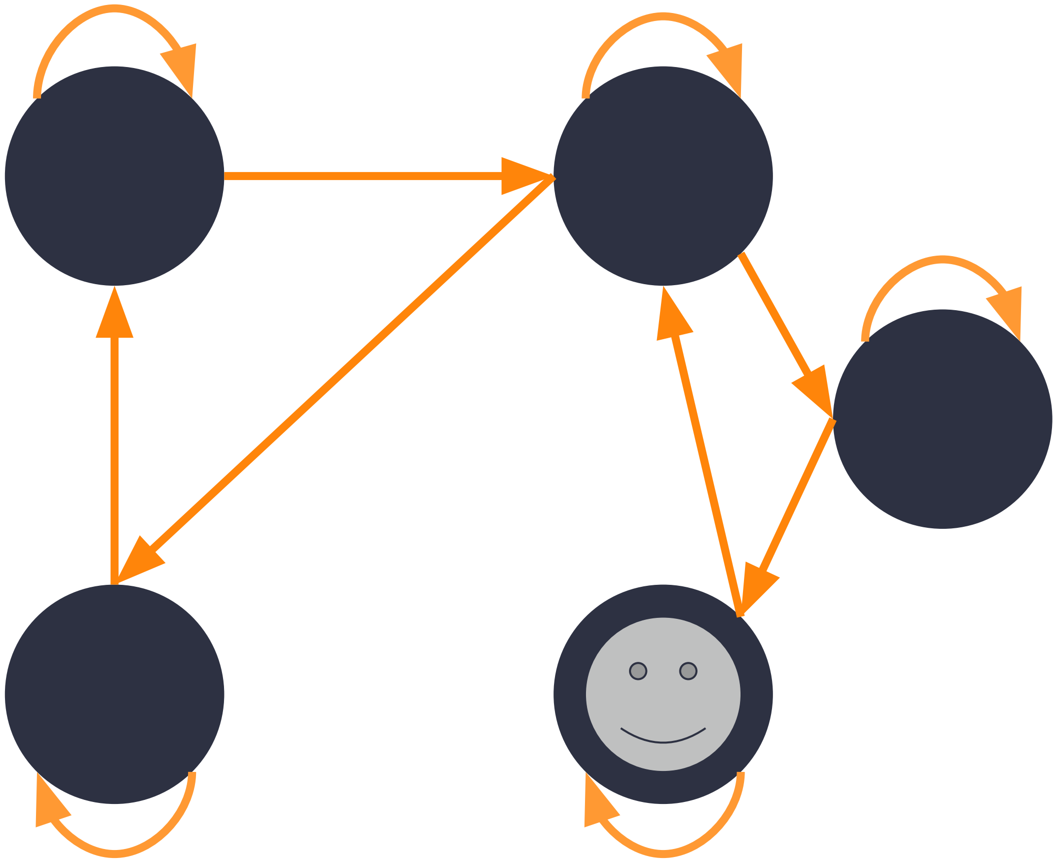

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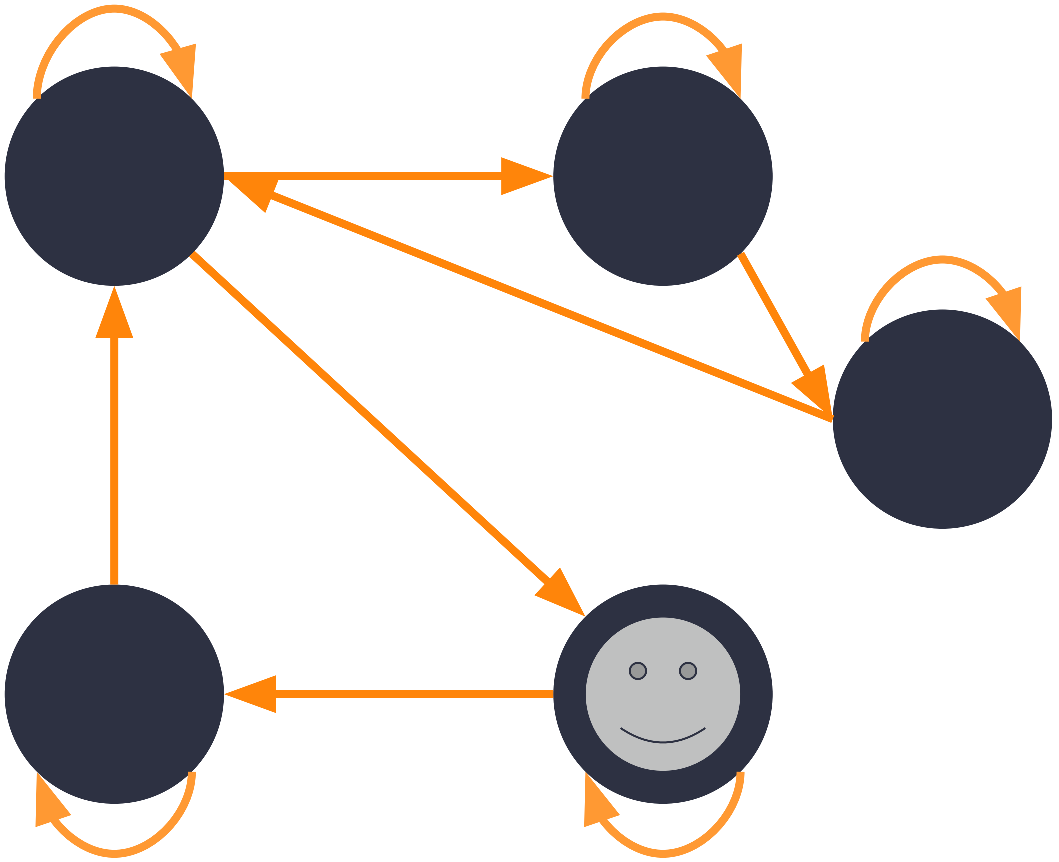

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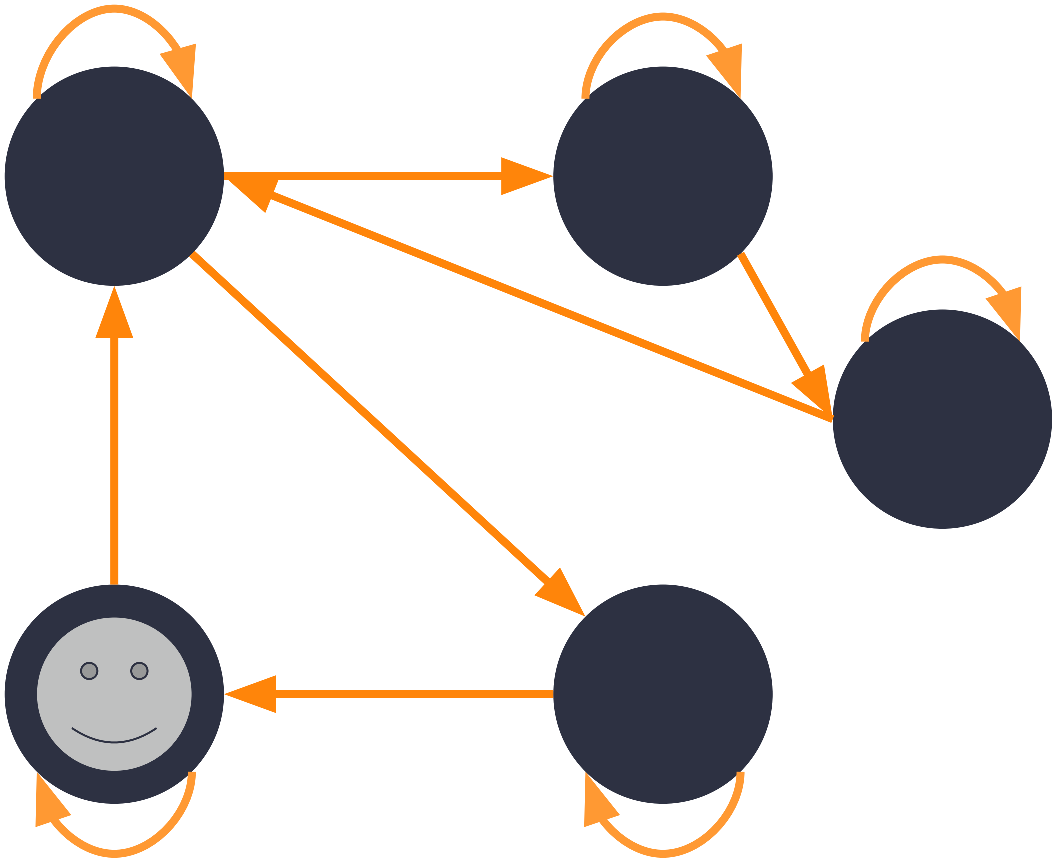
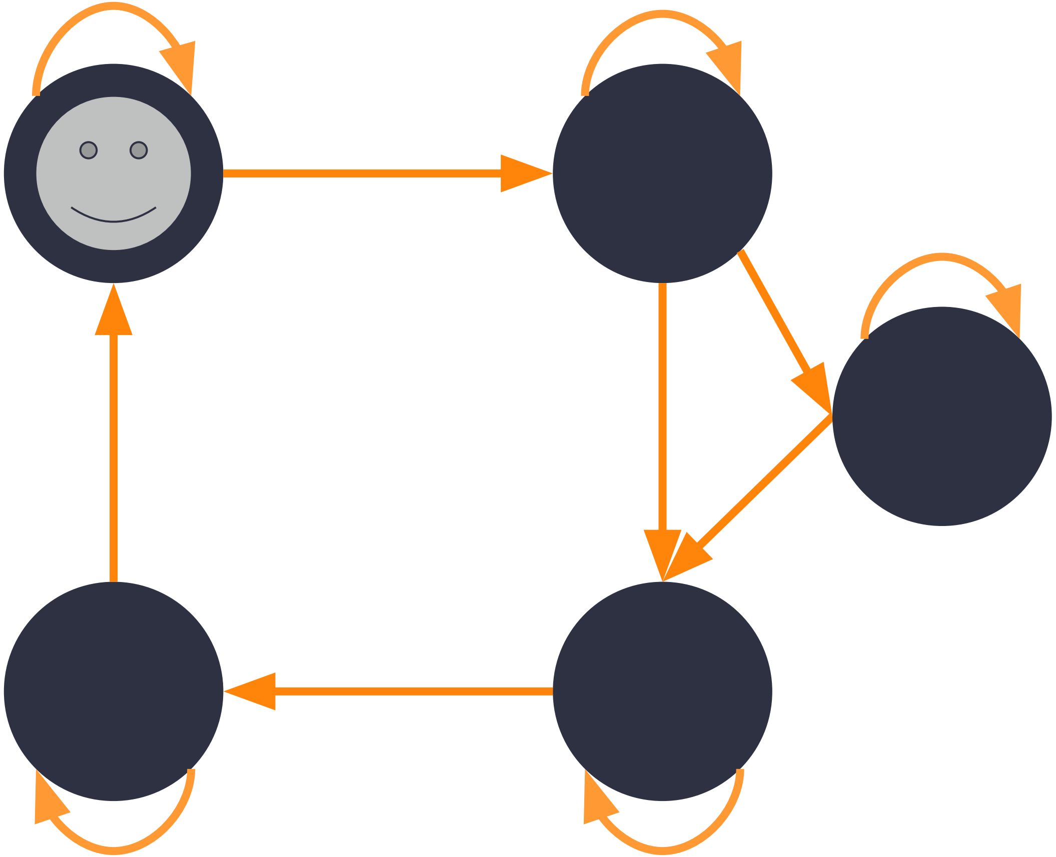
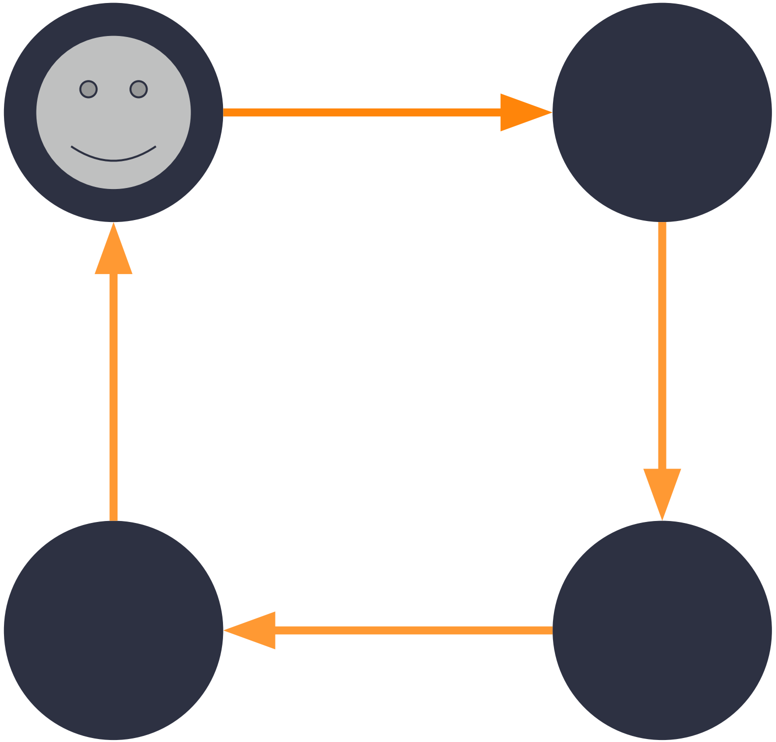


t = 6



t = 7

Recall



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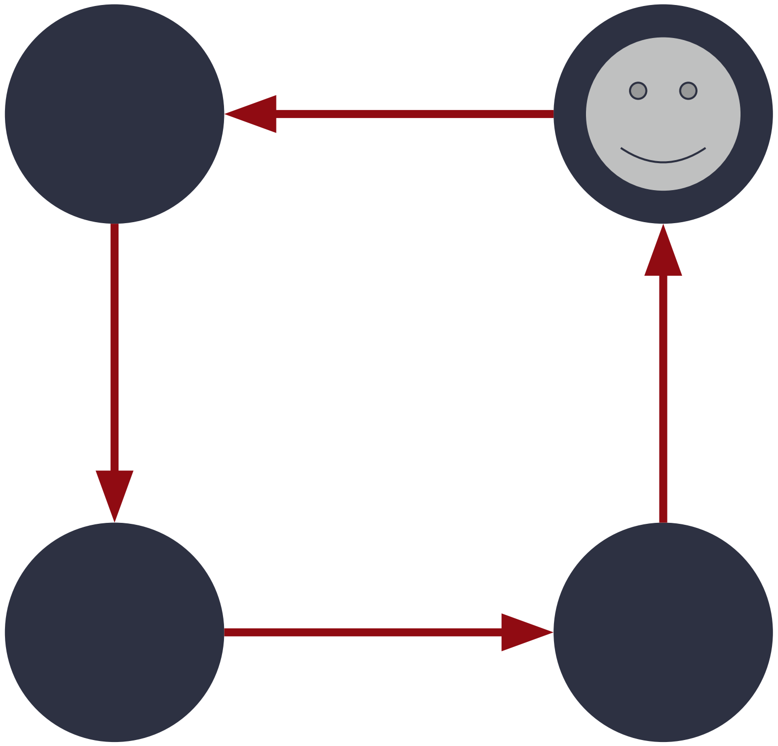

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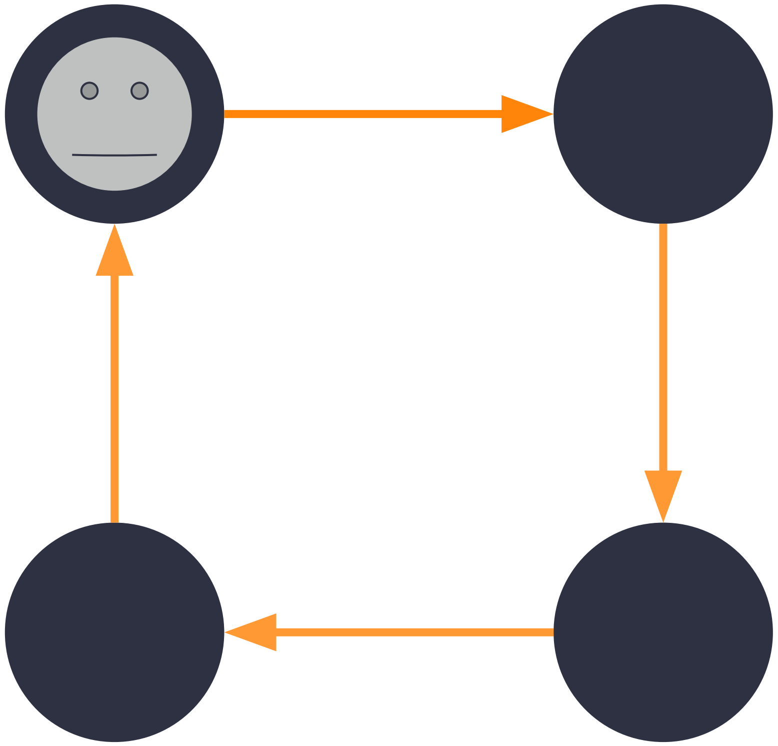

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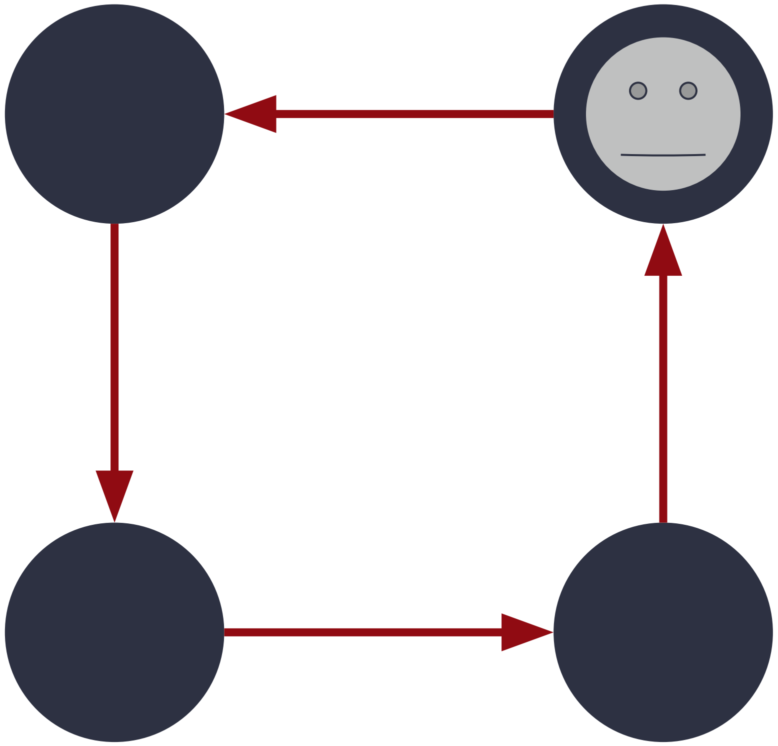
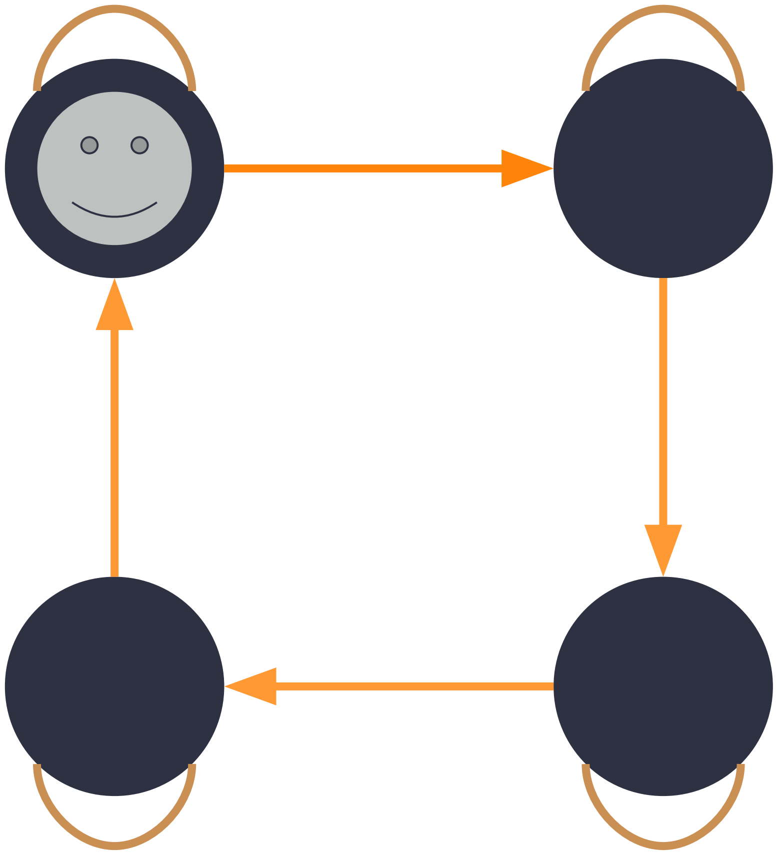


t = 4

Dynamic diameter is ∞

Fixed with Self-Loops



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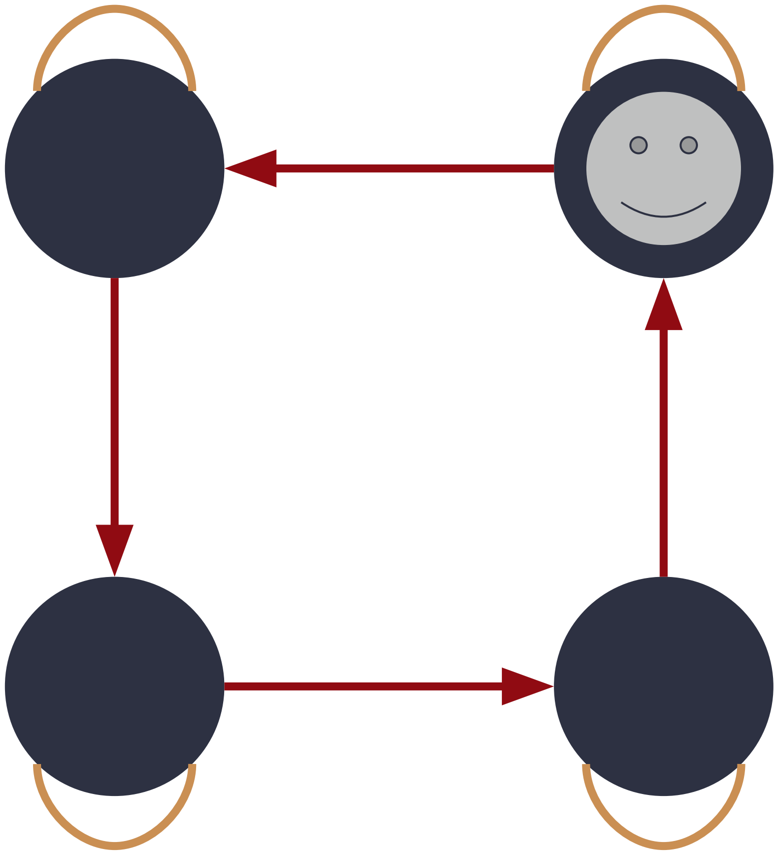

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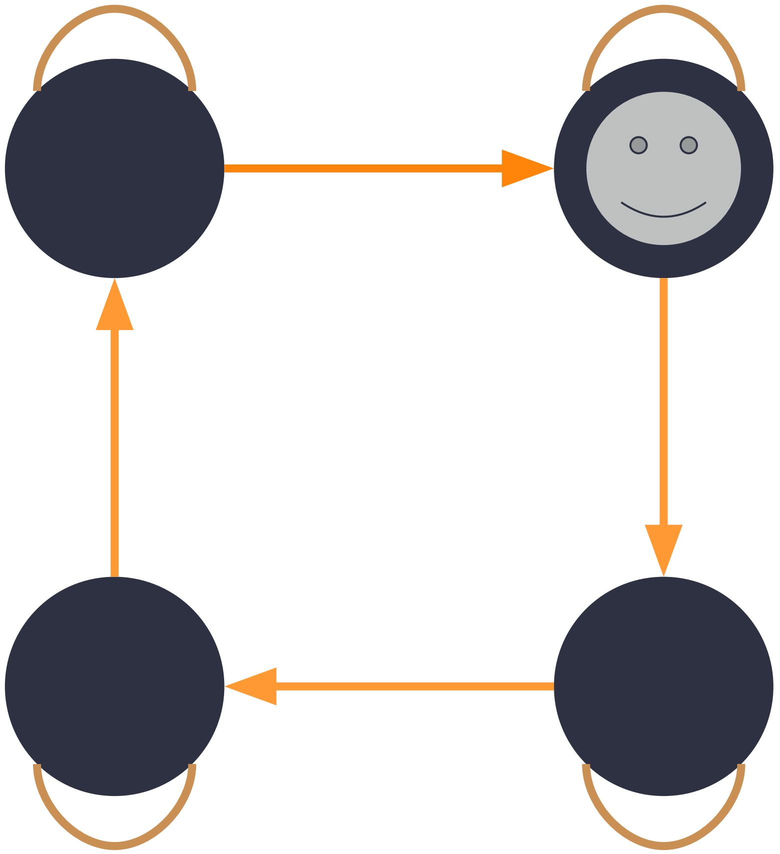

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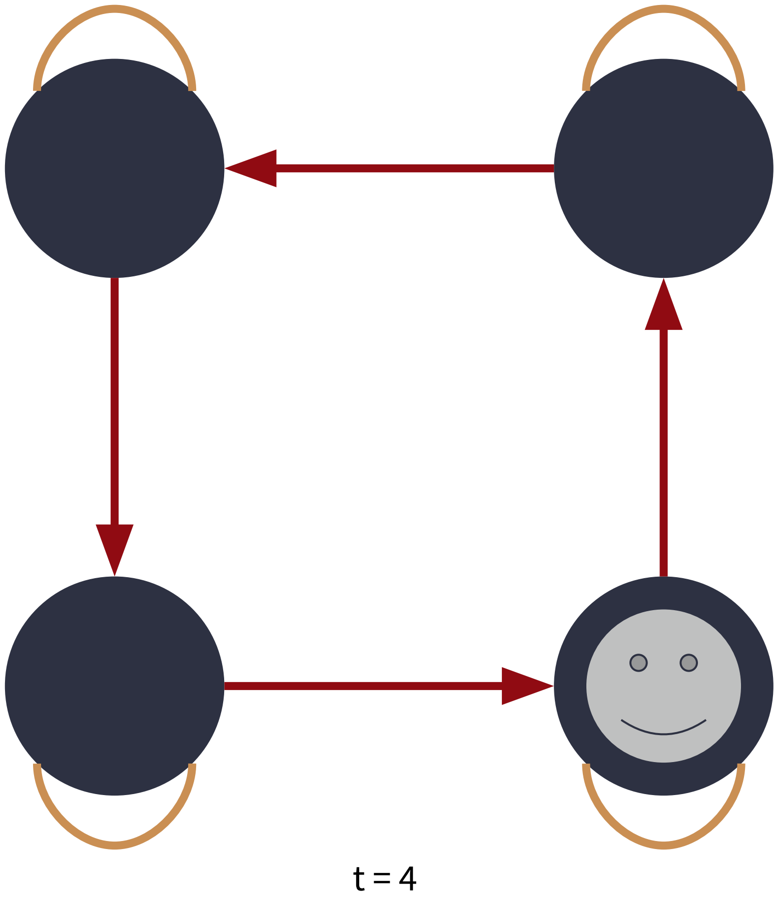
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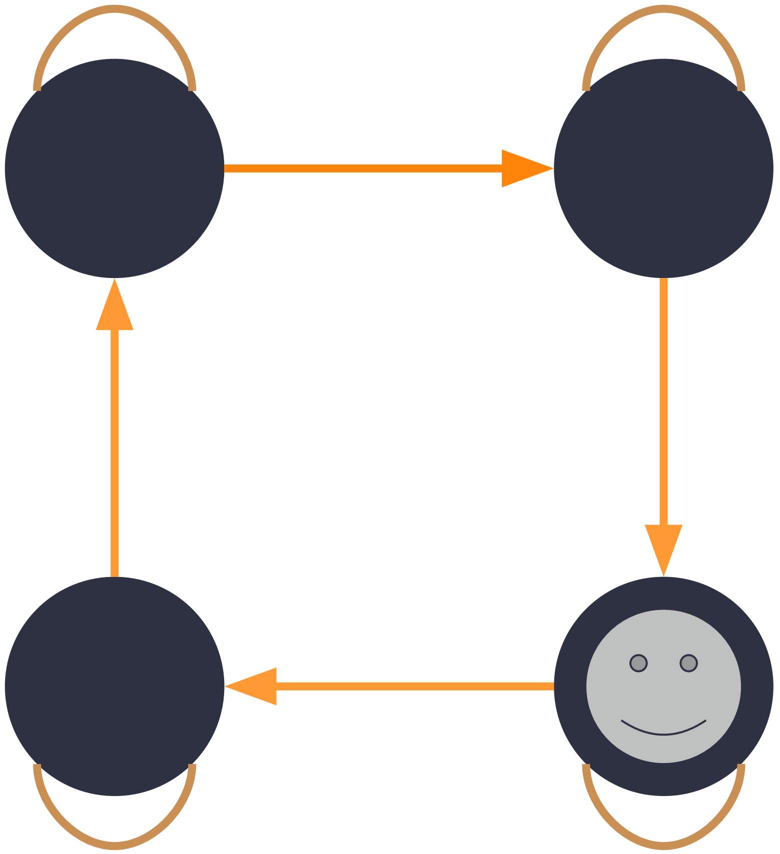

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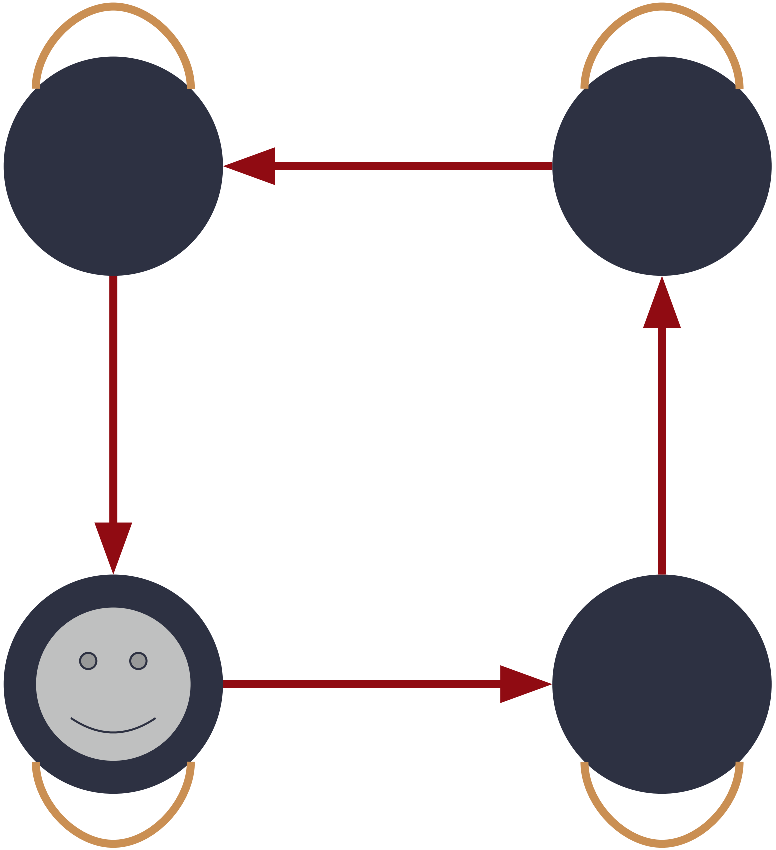

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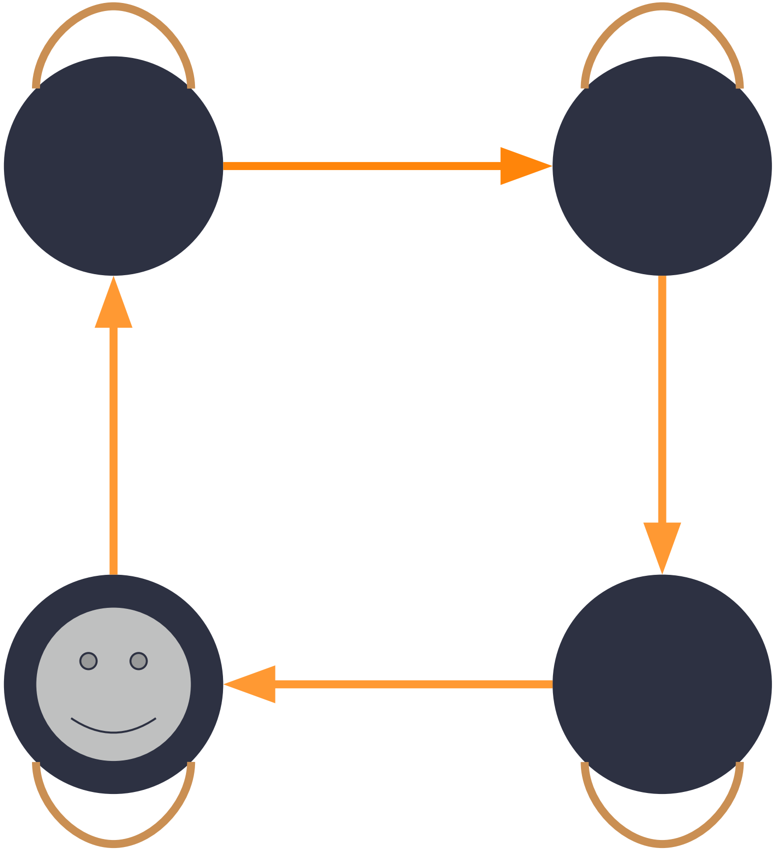

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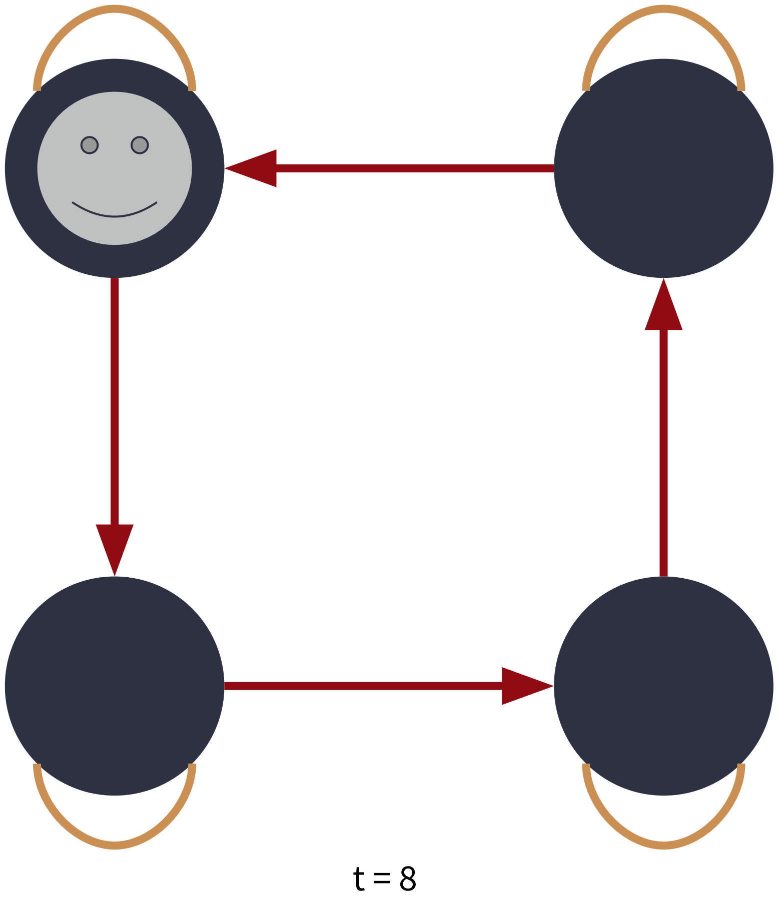
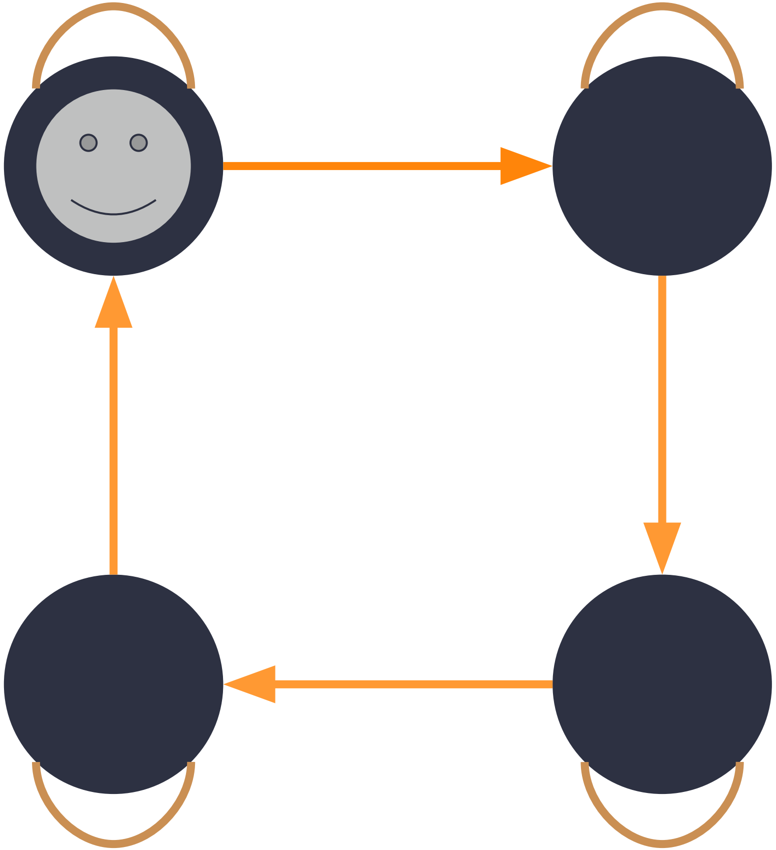

t = 8

Dynamic diameter is finite

Bound Achieved



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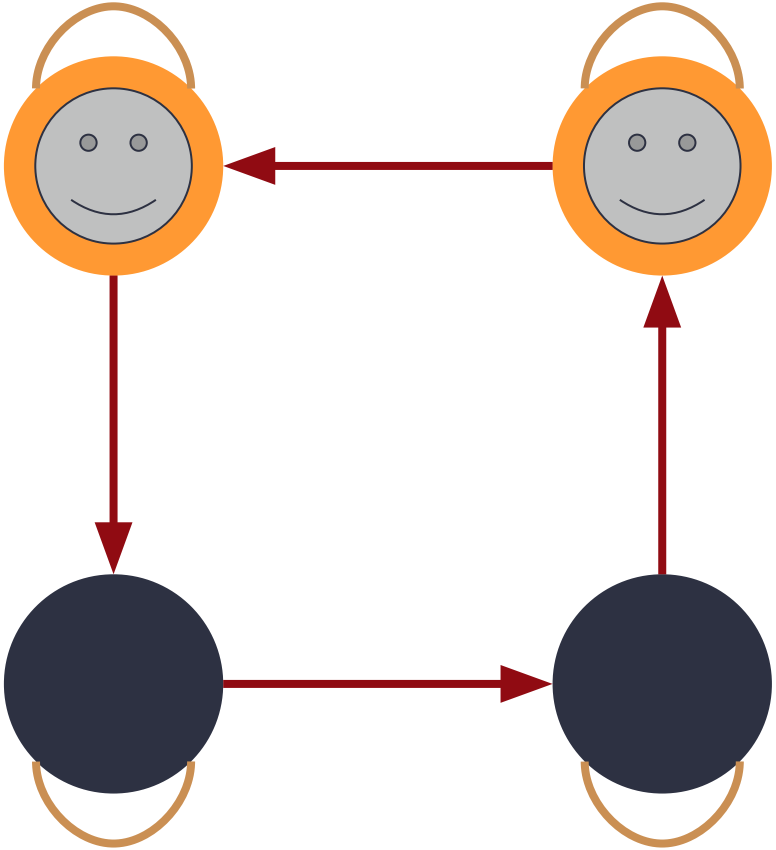

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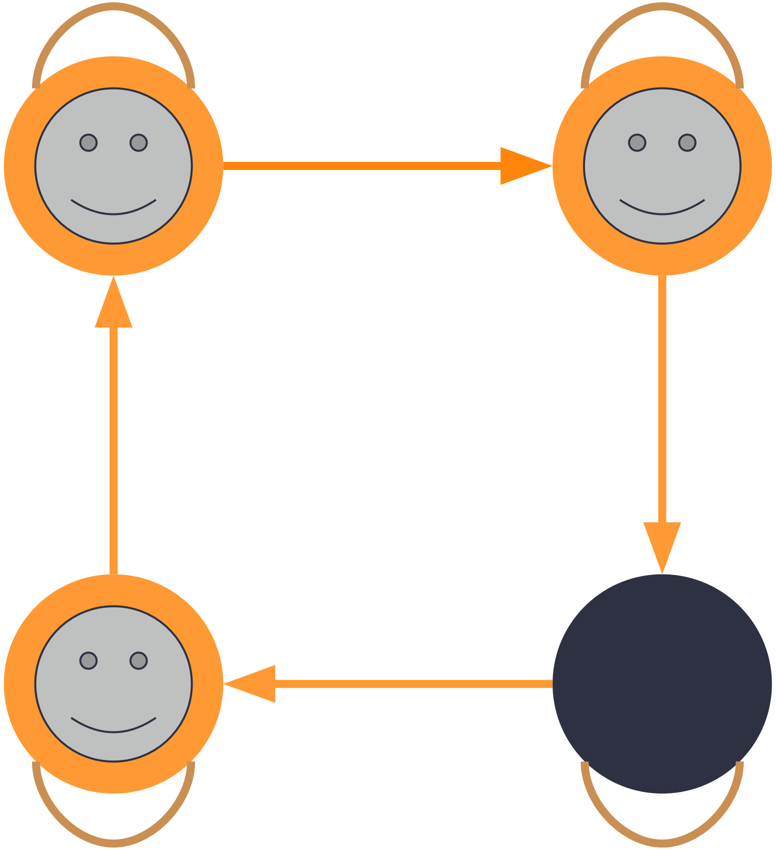

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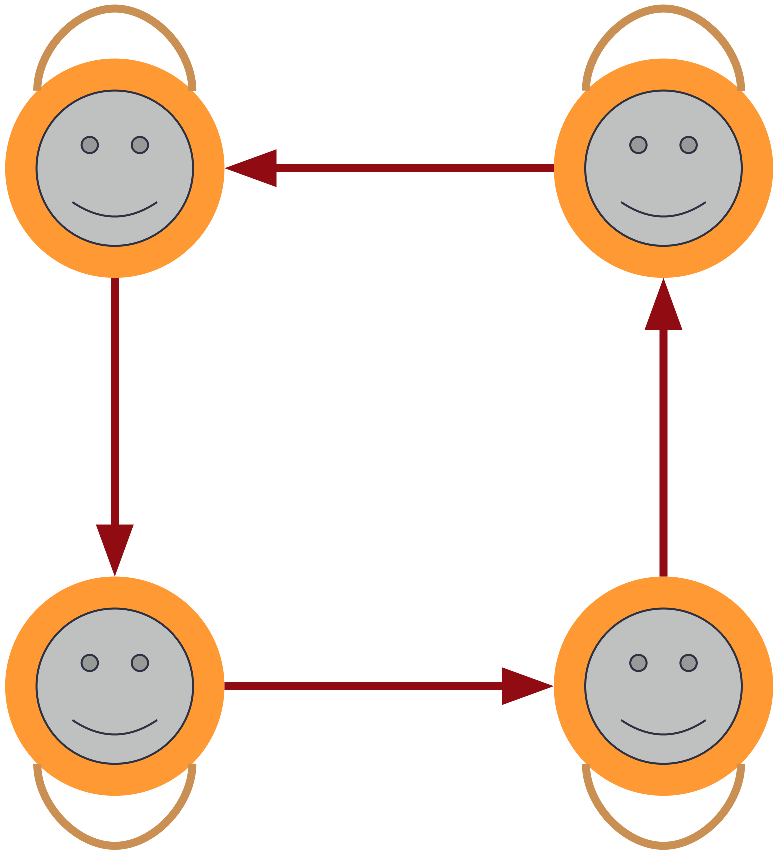


t = 4

Observation

Finding Conditions is Difficult

Idea

Stochastic Case

Observation: Force Edges to Move

- A particle can get pathologically “stuck”
- Require edges to change around
  - Ensure that each possible edge appears infinitely often?

Model: Dynamic Erdős-Rényi

- Fix edge probability $p \in (0, 1)$
- At every time step for every edge, flip a (biased) coin:
    - If heads, put the edge in
    - Otherwise, leave the edge out
- Note: edges across time are i.i.d. Bernoulli

<u>Observation</u>

Independence does not work

Proof: Independence => Disconnected

For each vertex at each timestep: probability of no outbound edges is $(1 - p)^n$, which is non-zero.

However ...

- Tweak: reflip all coins for a vertex if it has no outbound edges
    - Lose independence (a bit subtle)
- Based on simulations, conjecture: diameter is
    - constant if p constant
    - log n if p is (log n) / n

Observation

Self-loops are overpowered

Model: Dynamic Erdős-Rényi with Self-Loops

- Put in all self-loops
- Generate other edges (u, v) where $u \neq v$
  - Fix edge probability $p_{u,v} \in (0, 1)$
  - At every time step, flip a (biased) coin:
    - if heads, put in edge
    - otherwise, no edge

Proposition: Almost Surely Connected

1. Observation: every edge occurs infinitely often
2. By weak monotonicity, almost surely connected
3. Once connected, can never disconnect
4. Based on simulations, conjecture: diameter is
    a. constant if p constant
    b. log n if p is (log n) / n

Remaining Work

1. More rigorous treatment of non-self-loop case
2. Proof of proposed bounds
3. Additional models

Three Results

I. ~~Define dynamic analogs of static properties and relate them~~
II. Graph summarization: make representations more efficient
III. Connect to algebraic topology

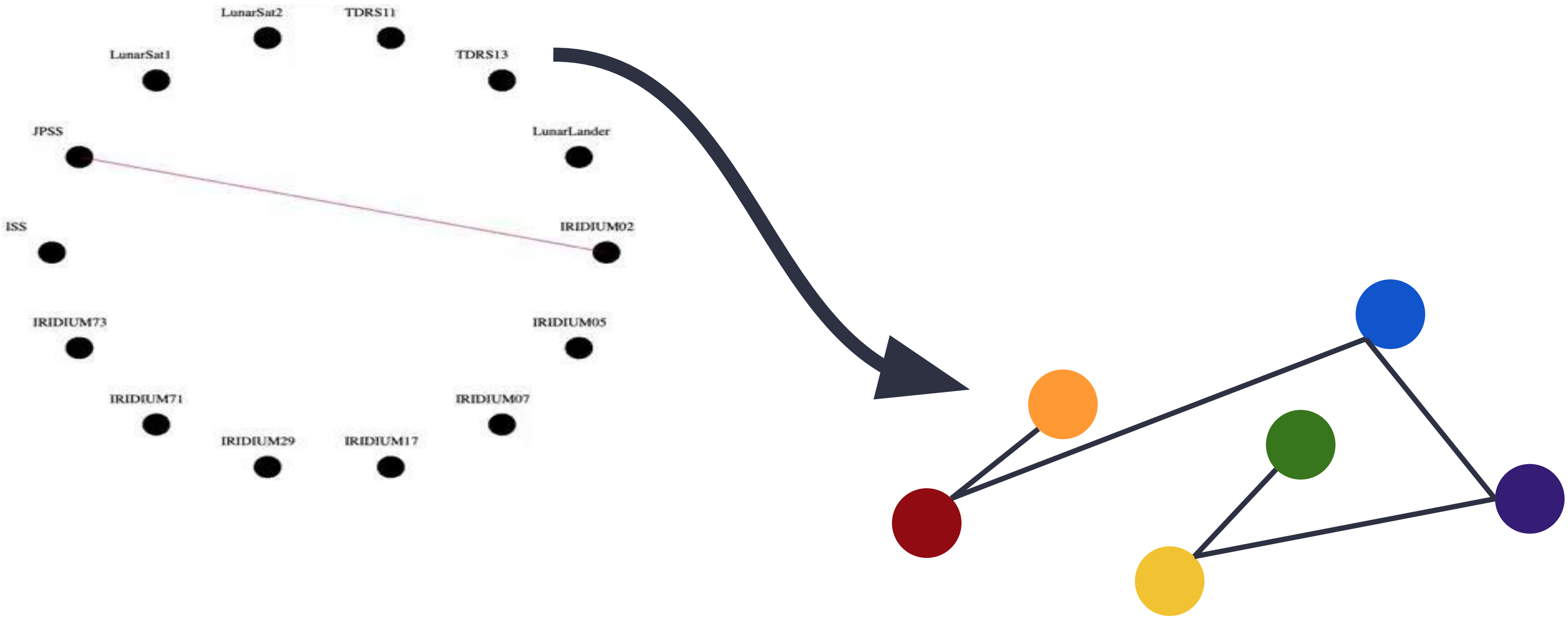
Graph Summarization

Idea

ν = n * t is bad

n is okay

Conversion


LunarSat2
TDRS11
LunarSat1
TDRS13
JPSS
LunarLander
ISS
IRIDIUM02
IRIDIUM73
IRIDIUM05
IRIDIUM71
IRIDIUM07
IRIDIUM29
IRIDIUM17

Strategies: Depends on Context

- Idea: each edge has time-based weight function
- Weight can represent
  - Capacity
  - Traversal time
  - Traversal speed
  - Bandwidth
  - etc.
- Inspired by centrality, take some “summary” of these functions

Shortest Path/Journey Participation

how often an edge appear in shortest paths or journeys

ex 1: Instant Path Summarization

1. Each edge has a (possibly infinite) cost as a function of time
2. Frozen in time, each path has a cost
3. Frozen in time, we can pick optimal paths if
    a. they have finite cost AND
    b. they have least cost

ex 1: Path Participation Function

$$\omega(P) = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^{\tau} \mathbf{1}_P(t)\, \mathrm{d}t$$

$$\bar{\omega}(e) = \sum_{P \in \mathcal{P} : e \in P} \omega(P)$$

ex 1. Weighted Static Graph

1. Started with: graph with time-dependent edge weight cost function
2. Ended with: graph with static edge weights
    a. Related to edge centrality measures
3. We can perform further analysis
    a. Katz centrality, etc.

ex 2. Traversal-Time Journey Summarization

1. Each edge has a (possibly infinite) traversal time as function of time
    a. If you start at time t, you will take w(t) units of time to traverse
2. Compute shortest journeys
3. Compute shortest-path participation for edges

ex 2. Traversal Time

$$\nu_e(t) = \frac{1}{w_e(t)}$$

The *velocity* across an edge

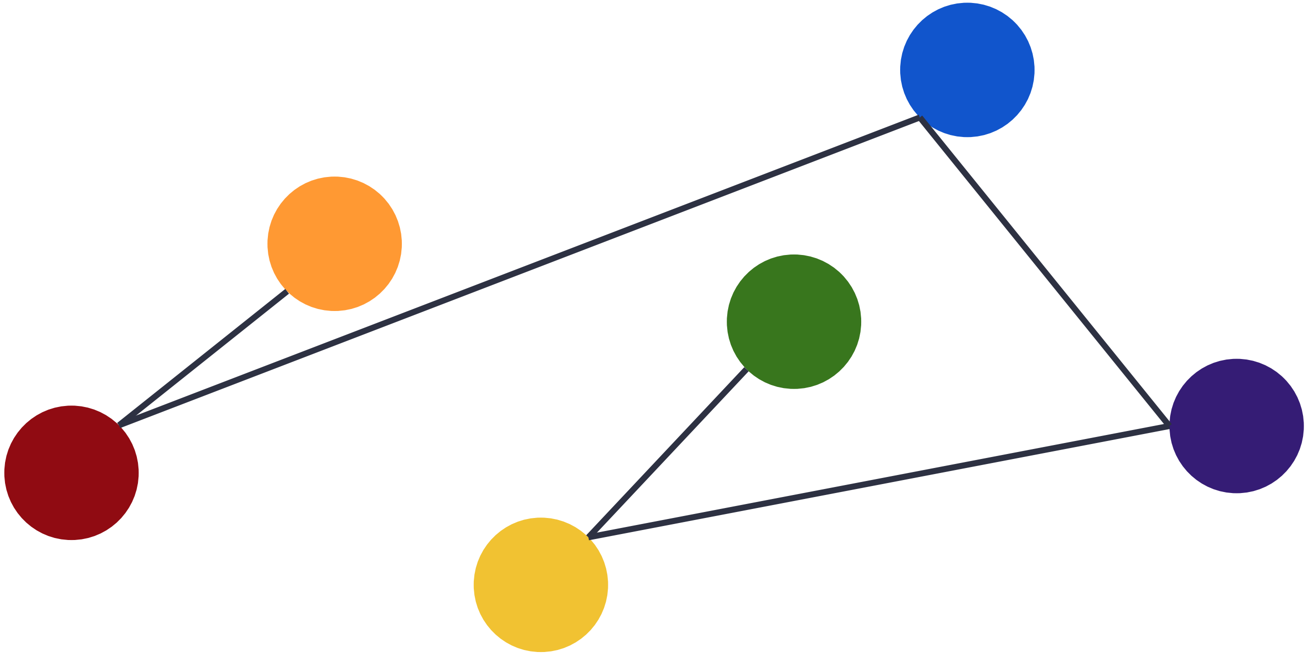
$$\mathrm{tcs}^s(e) = \left\{ x : \int_s^x \nu_e(t)\,\mathrm{d}t \geq 1 \right\}$$

The *traversal completion set*: at what time will we be done?

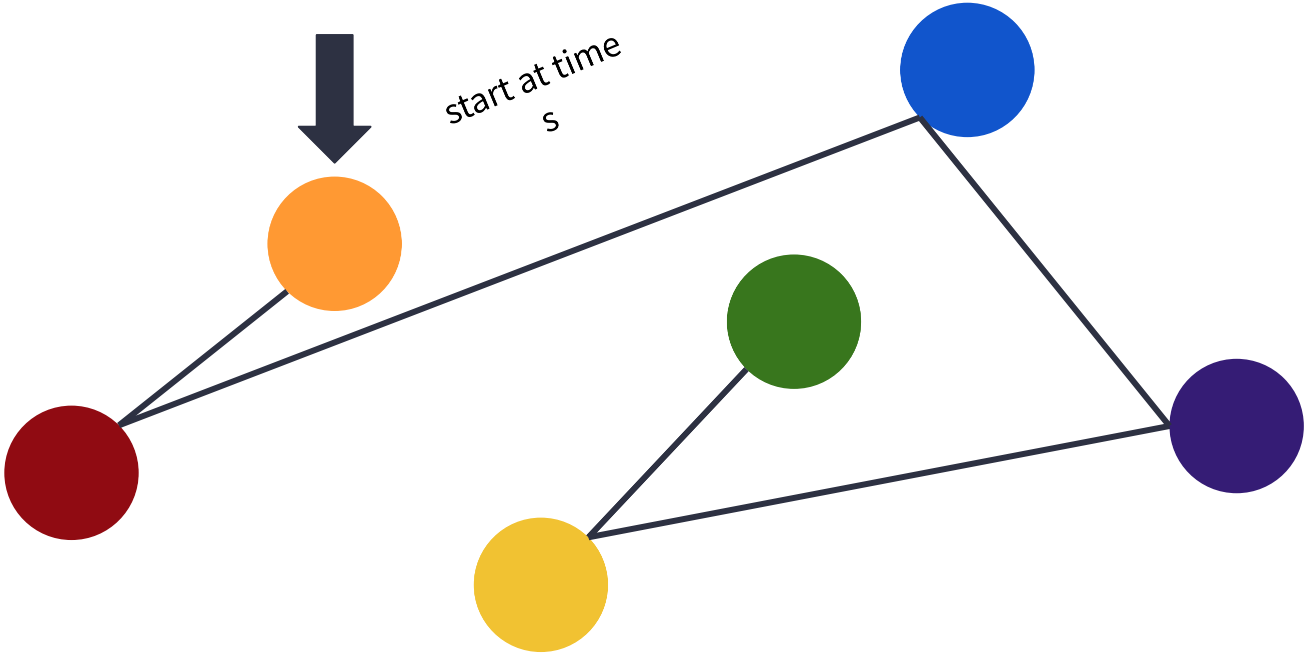
$$\mathrm{tt}^s(e) = \inf\{\mathrm{tcs}^s(e)\} - s$$

The *traversal time*: how long does it take to traverse the edge?

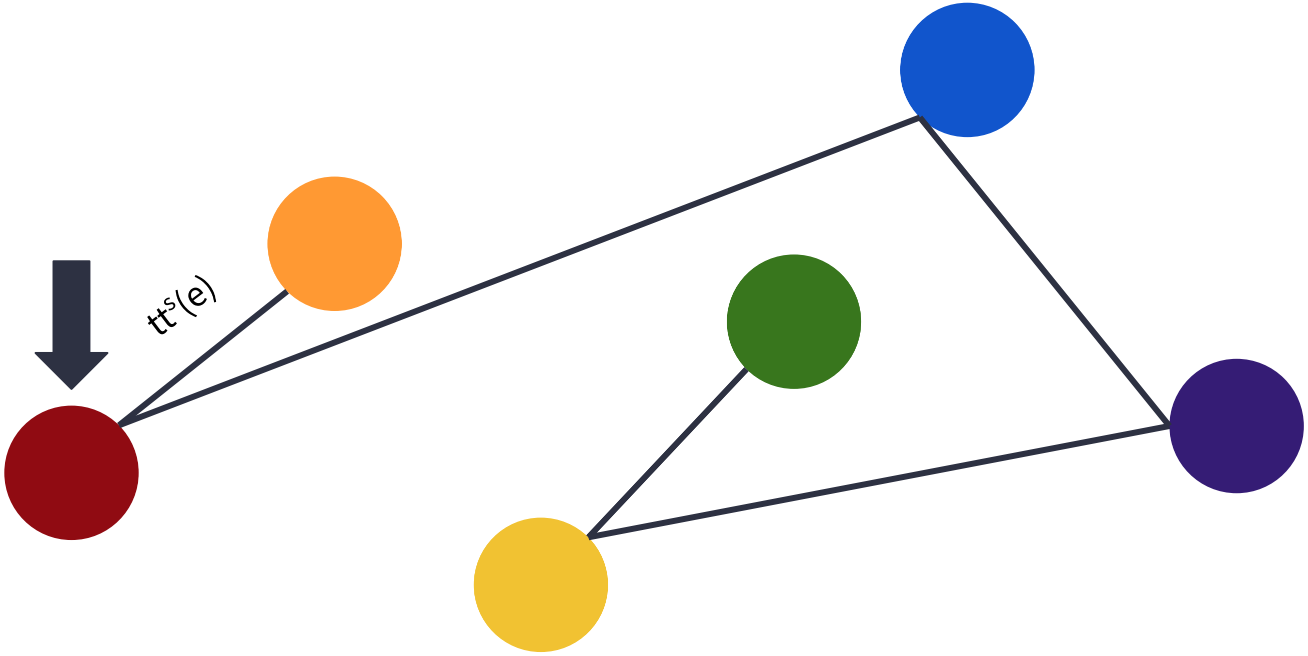
ex 2. Path Length



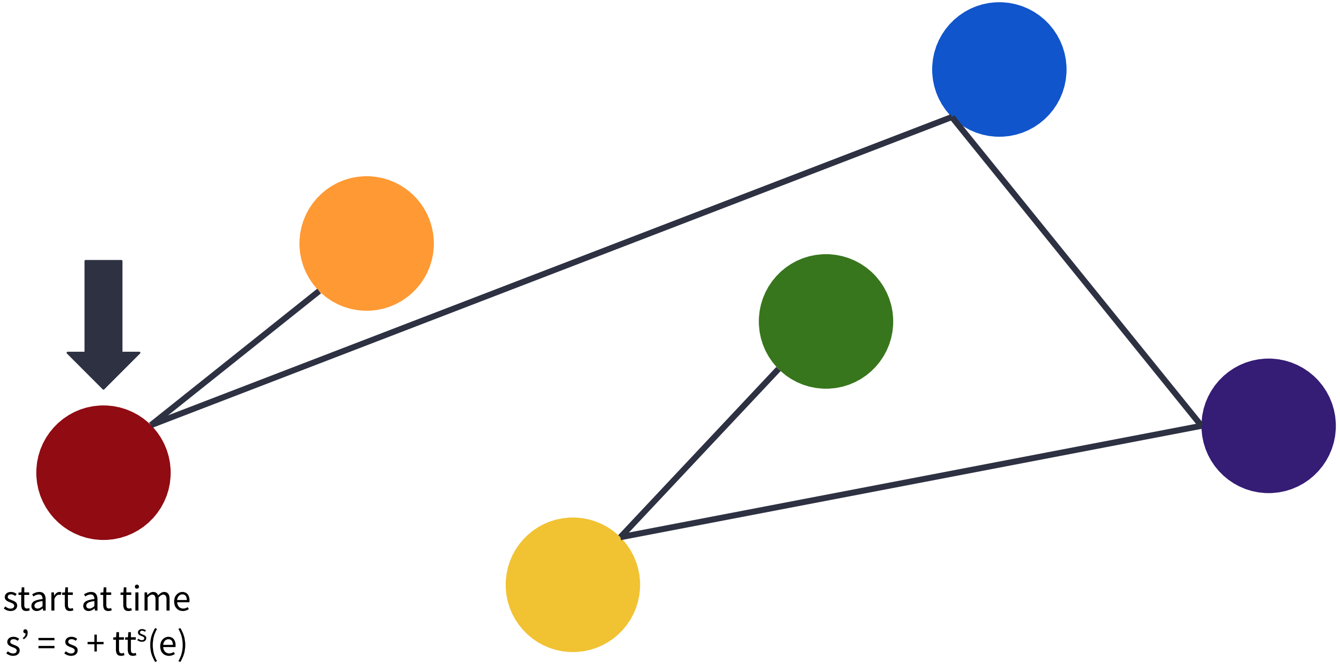
ex 2. Path Length



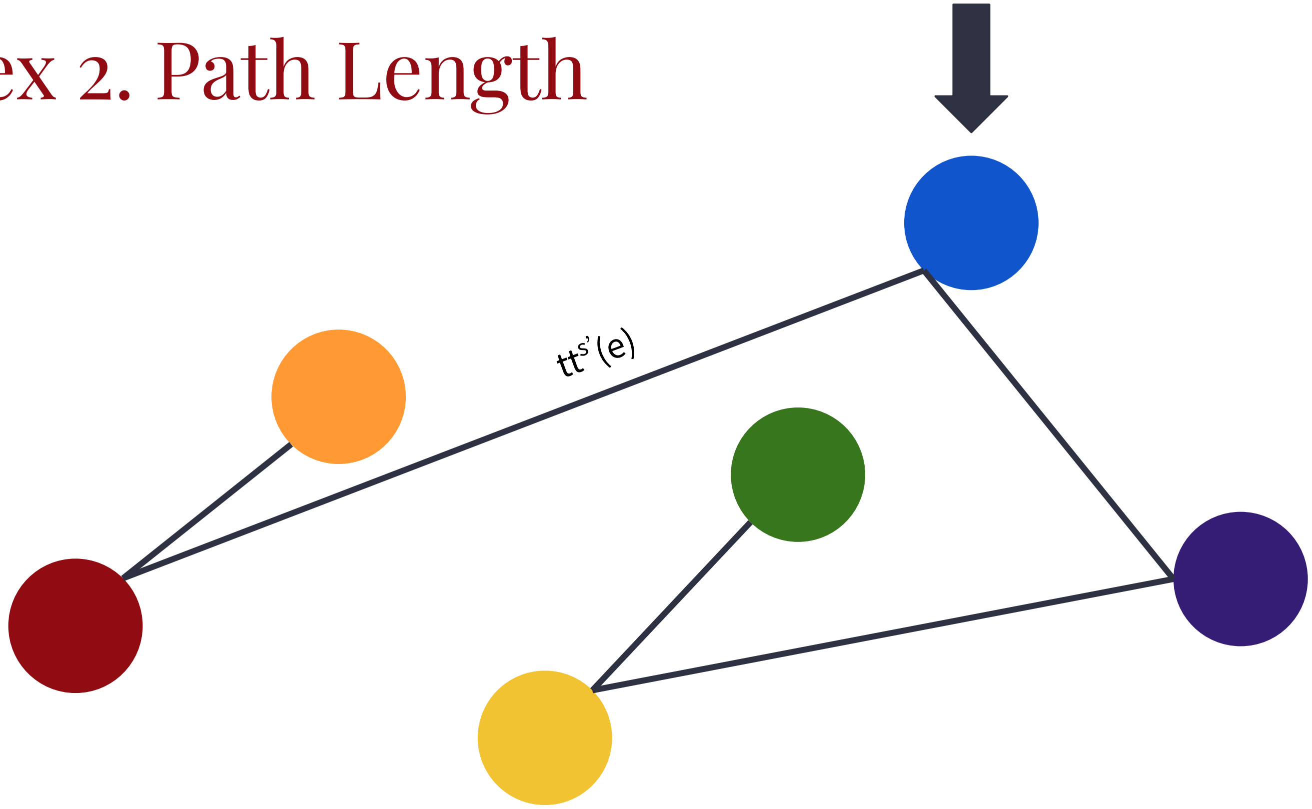
ex 2. Path Length



ex 2. Path Length



ex 2. Path Length



ex 2. Path Length

$$\mathrm{tt}^s(P)_1 = \mathrm{tt}^s(e_1)$$

$$\mathrm{tet}^s(P)_1 = \mathrm{tet}^s(e_1)$$

$$\mathrm{tt}^s(P)_2 = \mathrm{tt}^{\mathrm{tet}^s(P)_1}(e_2)$$

$$\mathrm{tet}^s(P)_2 = \mathrm{tet}^s(P)_1 + \mathrm{tt}^s(P)_2$$

$$\vdots$$

$$\mathrm{tt}^s(P)_k = \mathrm{tt}^{\mathrm{tet}^s(P)_{k-1}}(e_k)$$

$$\mathrm{tet}^s(P)_k = \mathrm{tet}^s(P)_{k-1} + \mathrm{tt}^s(P)_k$$

$$\mathrm{tt}^s(P) = \mathrm{tt}^s(P)_k$$

The ***path*** *traversal time*: how long does it take to traverse a path starting at time *s*?

ex 2. Final Step

1. We can define shortest paths at each time step if
   a. they have finite traversal time
   b. they have least traversal time
2. Then compute path participation as before

Summarization

Pros

- **Do expensive computation upfront**
- Compressed representation
- Extract relevant and meaningful information for application
- Creates a static graph, which we know more about than dynamic ones

Cons

- **Inherently lossy**
- Potentially expensive to compute
- Very specific to application
- Numerical issues

Extensions

1. Introduce additional summarization methods
2. Create a general (categorical?) framework for:
    a. Defining edge functions
    b. Summarization methods
    c. Analysis of summary graphs

Three Results

I. ~~Define dynamic analogs of static properties and relate them~~
II. ~~Graph summarization: make representations more efficient~~
III. Connect to algebraic topology

Algebraic Topology

Three Results

I. ~~Define dynamic analogs of static properties and relate them~~
II. ~~Graph summarization: make representations more efficient~~
III. ~~Connect to algebraic topology~~

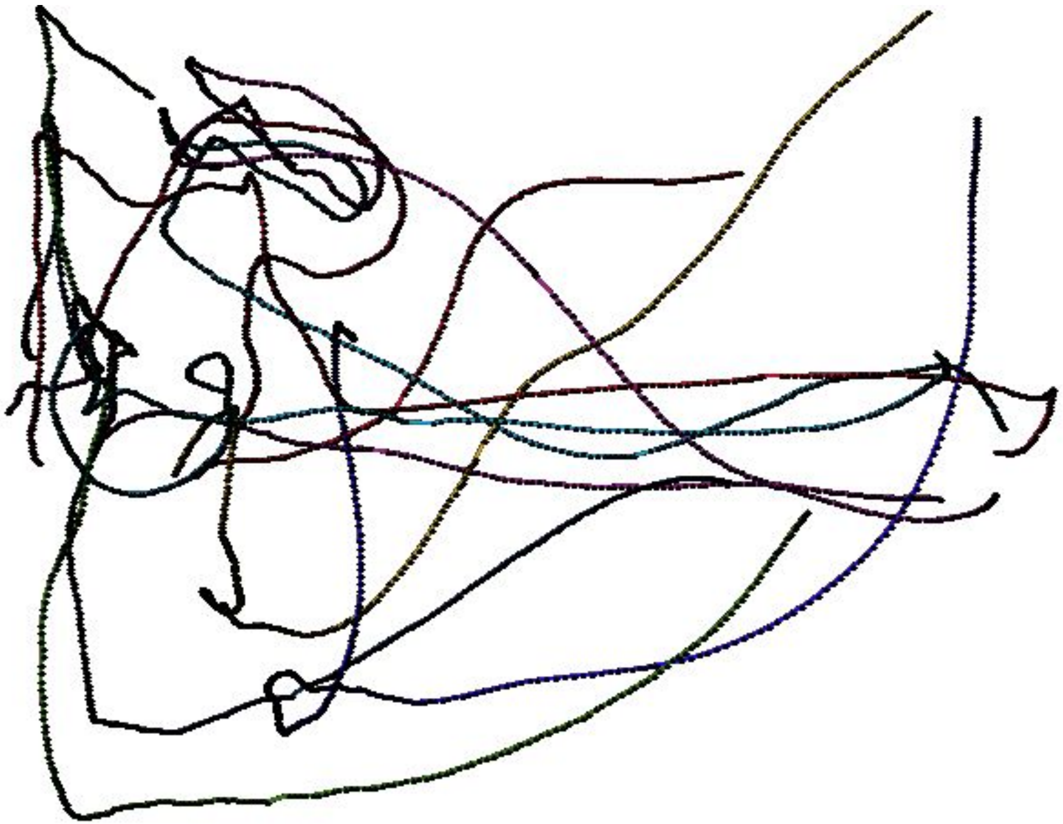
😎 Applications Abound 🧪

Dynamic Graph Projects

1. Viral spread across connected populations
    a. Rumors
    b. COVID-19
2. Basketball
    a. Using TDA
    b. Using ML
3. Space!
    a. Contact graph routing
    b. Tropical geometry
4. Others: animal clustering, transit, embryos, opinion dynamics

🤖 Machine Learning 🏀

Duke v. UNC (booooo!)
Multi-agent system (invasion sport)



Raw Trajectory Data

Dataset

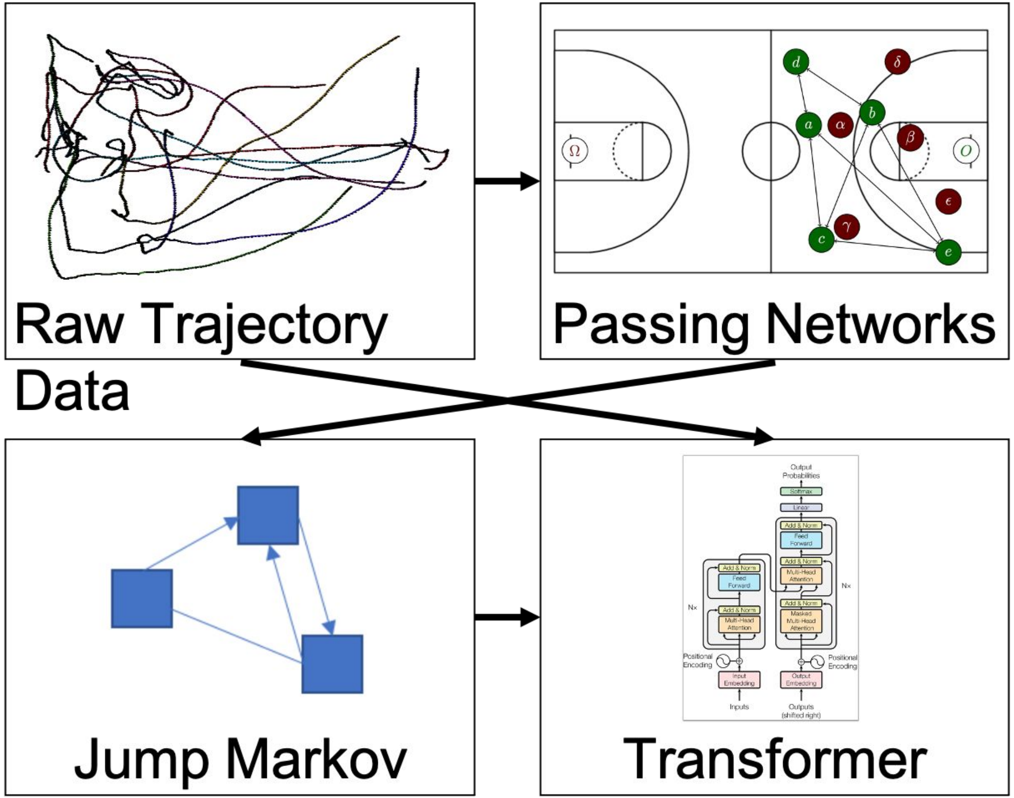
- (x, y)-coordinates of offense, defense, basketball
- 25 frames per second (40ms per frame)

Model Goals

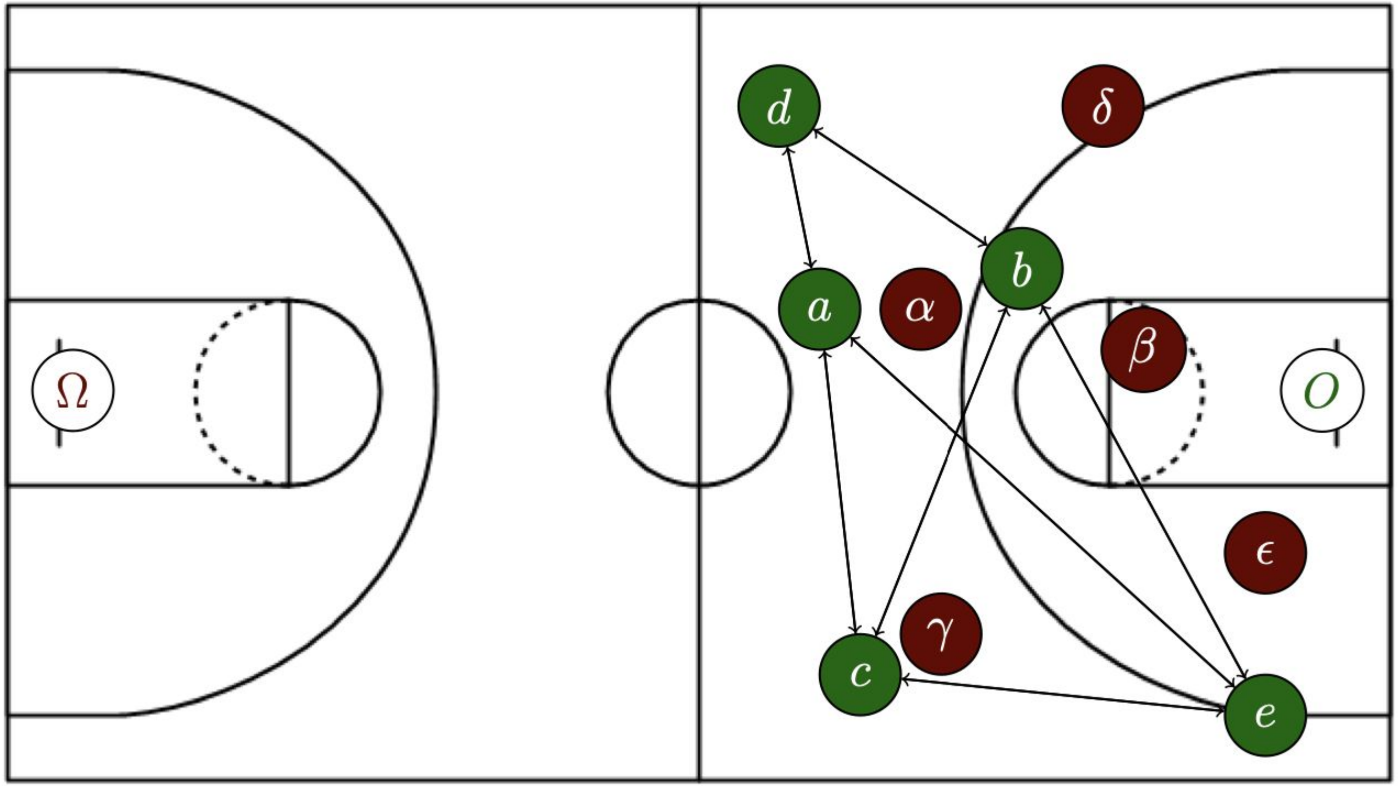
1. **Formation discovery**: a semantic understanding of the functional roles of players
2. **Compression and dimension reduction**: an efficient representation of a game, as player trajectory data is large and difficult to interpret
3. **Predictive power**: a mechanism for predicting trajectories of players
4. **Synthetic generation**: a tool for creating synthetic, but "realistic," data

Prior Work

1. **Trajectory Prediction**: related to predictive power and synthetic generation
2. **Role Discovery**: related to formation discovery
3. **Network Analysis**: related to high compression and dimension reduction



Model Pipeline



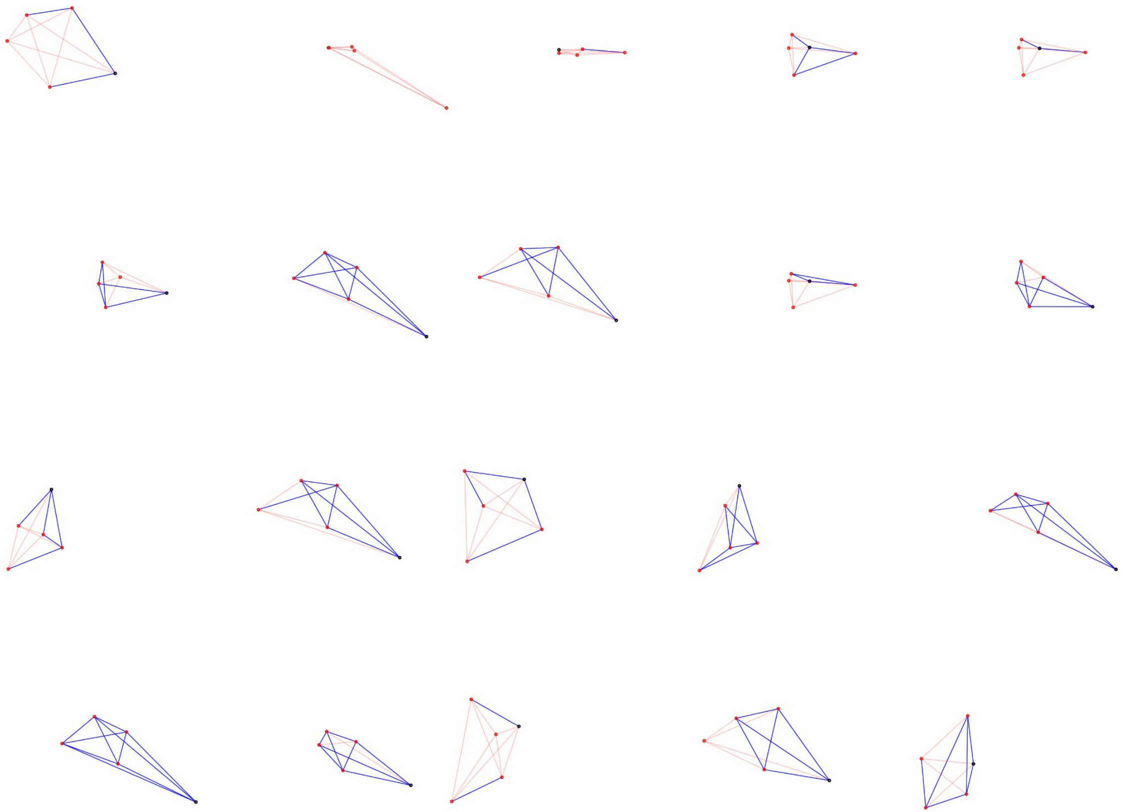
Step 1: Dynamic Passing Network

Observation

218 Graphs to Isomorphism

Step 2: Networks to Labels

1. Compute library of graphs seen in data
2. Assign each a unique label (frequency-based?)
3. Convert networks to labels



Passing Graphy Library

Interlude: Jump Markov

$$X(t) = E_{N(t)}$$

Interlude: Jump Markov

$$\boxed{X(t)} = E_{N(t)}$$

continuous-time *jump Markov process*

Interlude: Jump Markov

$$X(t) = E_{N(t)}$$

Poisson counting process

Interlude: Jump Markov

discrete-time *Markov chain*

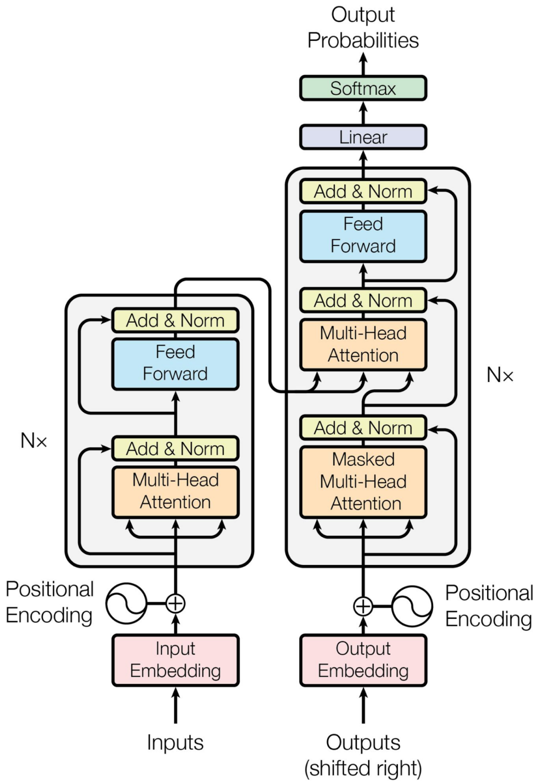
$$X(t) = \boxed{E}_{N(t)}$$

Interlude: Jump Markov

$$X(t) = E_{N(t)}$$

Idea: Semantic "Extraction"

1. Take library of passing graphs as tokens
2. Use NLP model to learn as a "language"
3. Good model: *Transformer*



Transformer Architecture

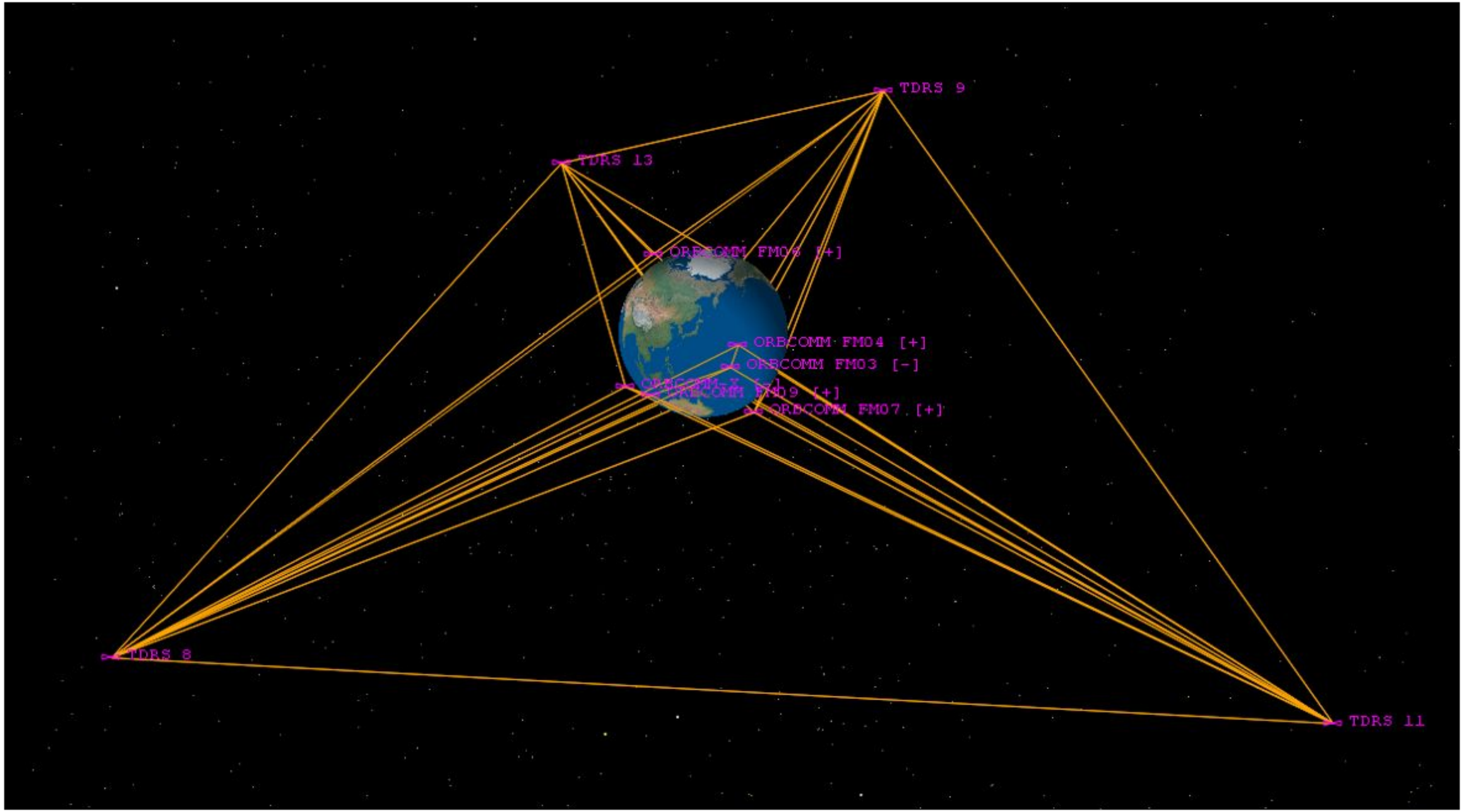
Experiment

- 40-10 prediction task
- Feed in graph data, along with base position data
- Predict trajectories
- Compare against true trajectories with MSE

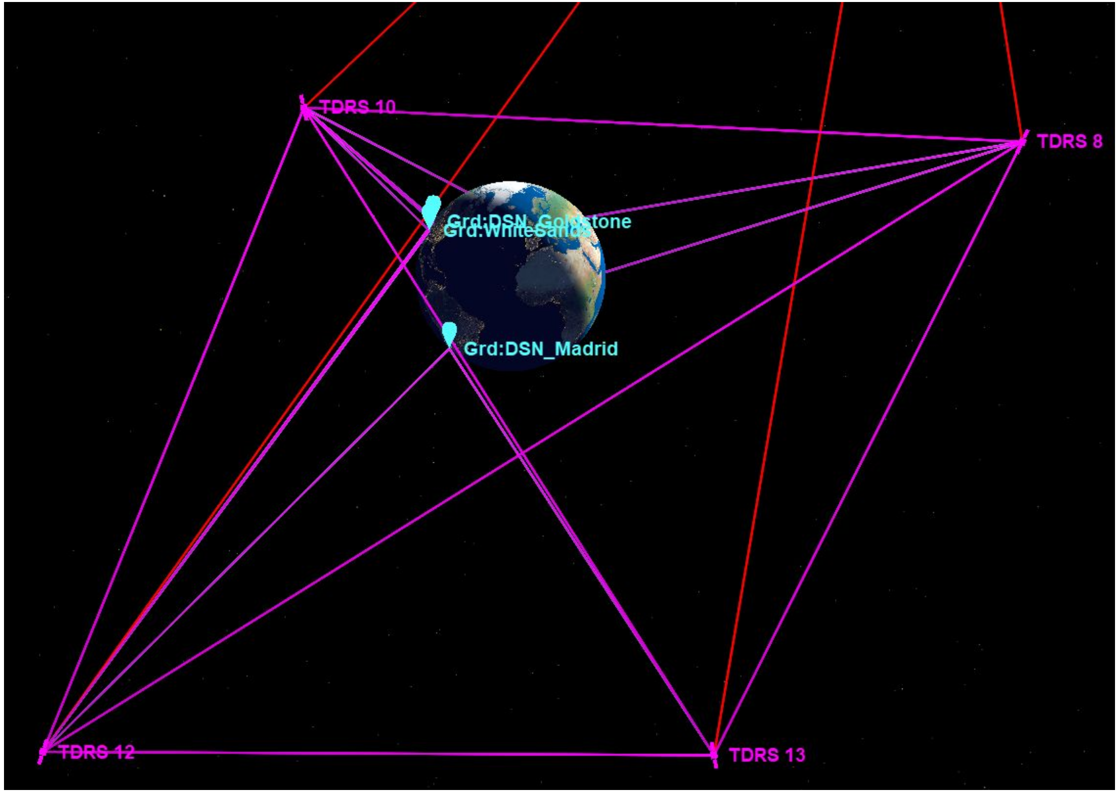
66%

reduction in loss against benchmark
(40–10 trajectory prediction task)

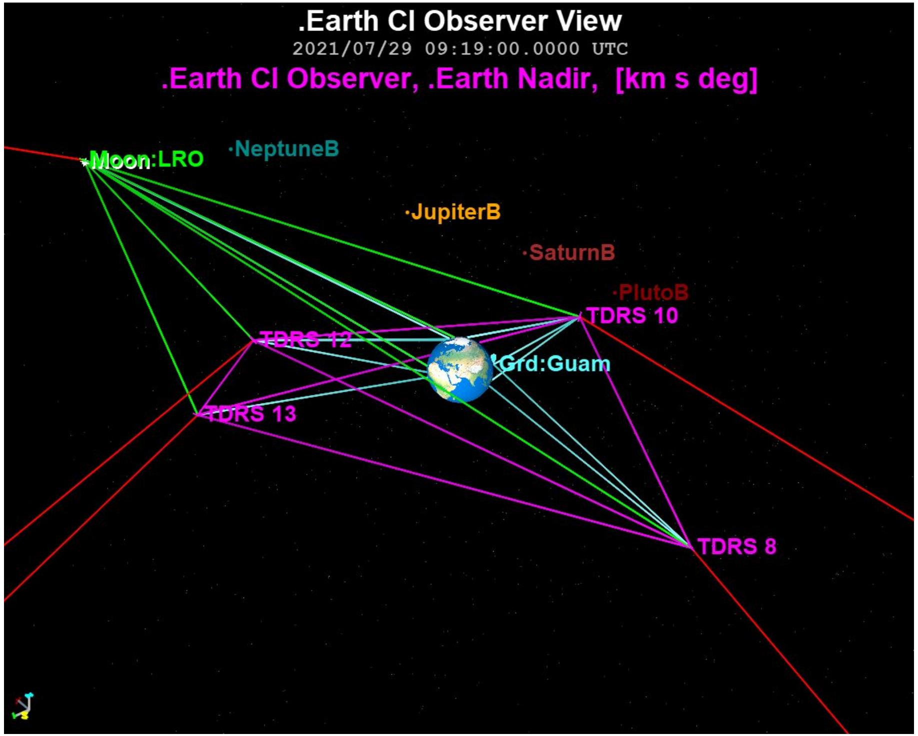
🚀 Space 🌕


TDRS 9
TDRS 13
ORBCOMM FM04 [+]
ORBCOMM FM03 [-]
ORBCOMM FM07 [+]
TDRS 8
TDRS 11

Satellites



More Satellites



Even *More* Satellites

Contact Graph Routing

1. Satellites, ground stations, etc. moving in space
   a. Data tells us when two will make “contact” with each other
   b. Format is a “contact graph”
   c. Graph is dynamic and periodic (idealized)
2. Task: route information across network
   a. High latency
   b. Changing bandwidth + speed
   c. Low storage
   d. High error rate

Current Approach

Send information in packets and repeat 10 times based on known orbit schedule.

Goal

Can we do better?

Defining “Better”

1. Increase bandwidth (worst case is 2kbps)
2. Decrease latency (limited by speed of light)
    a. 1.3 seconds to Moon
    b. ~25 minutes to Mars
    c. 5.5 hours to Pluto
3. Decrease error rate
4. Add features
    a. Packet prioritization
    b. Broadcasting

Defining "Better"

1. **Increase bandwidth (worst case is 2kbps)**
2. Decrease latency (limited by speed of light)
    a. 1.3 seconds to Moon
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    c. 5.5 hours to Pluto
3. Decrease error rate
4. Add features
    a. **Packet prioritization**
    b. Broadcasting

Generalization of Jowsig

1. Start with weighted digraph
2. Perform Dijkstra’s repeatedly
3. Generate shortest path trees
4. Repeat until we reach fixed point

🔮 Future Work 🔬

Theory

1. Further extensions of static properties and their relationship to their dynamic counterparts
2. Analysis of the stochastic setting
3. General framework for summarization
4. Clustering (spatiotemporal *k*-means)
5. Generalized optimal routing
6. Periodic systems

Applications

1. Animal herding behavior
2. General Transit Feed Specification (GTFS)
3. Twitter data
4. Additional satellite data

In conclusion …

https://xkcd.com/1403/

🙋 Q&A 🙃